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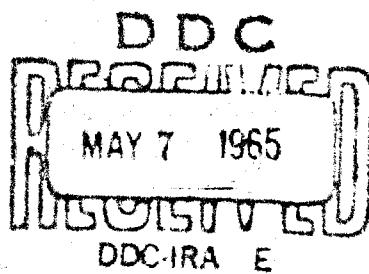


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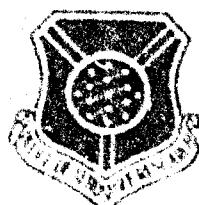
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The Perfectly Coupled and Shunt-  
Augmented T Two-Port

KURT H. HAASE



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DATA SCIENCES LABORATORY PROJECT 5628

**AIR FORCE CAMBRIDGE RESEARCH LABORATORIES**

L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

## **The Perfectly Coupled and Shunt-Augmented T Two-Port**

**KURT H. HAASE**

**OFFICE OF AEROSPACE RESEARCH  
United States Air Force**



## **Abstract**

In this report a new two-port structure, referred to as a "Perfectly Coupled and Shunt-Augmented T", is defined and its properties described. The classical Brune two-port can be recognized as a particular example of this general class of two-ports. There are three feasible types (A, B, and C) of perfectly coupled and shunt-augmented T's. A tandem of two matched T's of type AC or BC is equivalent to a lattice two-port. Suitable impedances can be transposed from one port to the other over the T whereby only the magnitudes of its elements are changed. The dual of this T is the "Perfectly Coupled and Series-Augmented Pi". The discussion of these new kinds of two-ports is supplemented by ten numerical examples.

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## Foreword

In 1963 Fusachika Miyata published an article concerned with the realization of certain one-port impedance functions in the form of a lattice two-port. He has shown that under certain conditions the impedance function can be developed as a tandem of two classical sections (Brune, 1931), and that under further conditions this tandem is equivalent to a lattice.

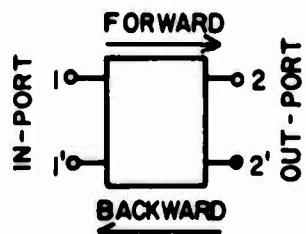
The author of the present report has discovered that Miyata's findings, and with them the classical Brune concept, can be generalized. For this purpose he defines and uses a class of two-port sections which he refers to as "Perfectly Coupled and Shunt-Augmented T Sections" (abbreviated in the report by *pcsa T*). This two-port section represents an entirely new concept. It is the aim of this report to show and to prove some of its properties in Part I. In Part II we will present some numerical examples. Although the *pcsa T* can be considered as a fundamental structure in one-port realizations, we will not discuss how the section can be derived from a given impedance function; this is a major problem and will be discussed in another report.

## Two-Port Terminology

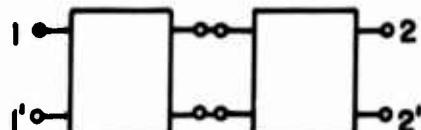
In this report we are only concerned with passive two-ports; these are two-ports which incorporate only resistances, capacitances, and inductances with and without mutual coupling (transformers). A two-port has two pairs of terminals 1 and 1', and 2 and 2' as shown in the figure on the left. The terminal pair 1 and 1' is referred to as the in-port, the terminal pair 2 and 2' as the out-port. This terminology seems to be reasonable since a passive two-port traditionally is supposed to be energized at the left-side in-port. For this reason we will also define the direction from the in-port to the out-port as the forward direction, and the opposite direction as the backward direction.

A tandem is a chain structure of at least two two-port sections such that the out-port of one section is connected with the in-port of the next section in the forward direction as shown in the figure on the right.

An impedance branch that is connected to a port is referred to as an in-port impedance or as an out-port impedance respectively.



Orientation at a Two-Port



A Tandem of Two Two-Ports

## THE PERFECTLY COUPLED AND SHUNT-AUGMENTED T TWO-PORT

### Part I The Definition and the Properties of a Perfectly Coupled and Shunt-Augmented T

#### I. DEFINITIONS AND CHAIN MATRIX

##### 1.1 The Definition of a Perfectly Coupled T

Assume that the branches of the T section shown in Figure 1 have the impedances  $U$ ,  $V$ , and  $W$  such that

$$U = u \cdot \phi(s) , \quad (1a)$$

$$V = v \cdot \phi(s) , \quad (1b)$$

$$W = w \cdot \phi(s) , \quad (1c)$$

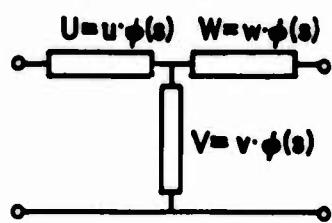


Figure 1. A Perfectly Coupled T Section

In Eqs. (1a, b, c) are the  $u$ ,  $v$ , and  $w$  real constants, which are not necessarily positive. The notation  $\phi(s)$  in these equations represents a positive real and normalized frequency function. We understand the normalization in such a way that if we write function as a quotient of two polynomials  $p(s)$  and  $q(s)$  as in Eq. (2), the coefficients associated

with the highest powers of  $s$  are 1 in the numerator and in the denominator and the factor before the fraction is 1:

$$\phi(s) = \frac{1 \cdot s^m + a_{m-1} s^{m-1} + \dots}{1 \cdot s^n + b_{n-1} s^{n-1} + \dots} , \quad (2)$$

where the degrees are either  $m = n$ , or  $m = n \pm 1$ .

As usual, the frequency variable  $s$  is  $s = \sigma + j\omega$  (with  $\sigma = 0$ ).

Since  $U$ ,  $V$ , and  $W$  denote impedances, we contribute the dimension "impedance" on the right sides of Eqs. (1a, b, c) to the constants  $u$ ,  $v$ , and  $w$ , and we say that  $u$ ,  $v$ , and  $w$  are constants of impedance character and the  $\phi(s)$  is a scalar frequency function. We have to admit that this yields to some controversy when  $\phi(s) = s$  or if  $\phi(s) = 1/s$ . In this event, for example,  $u \cdot s$  is an impedance; but  $u$  is considered as an impedance character rather than as an inductance: similarly in  $u/s$ ,  $u$  is also considered an impedance character rather than an inverse capacitance. But in connection with all frequency functions of higher order it is convenient to consider these functions as scalars.

Assume now that the constants are interrelated by the equation

$$1/u + 1/v + 1/w = 0 . \quad (3)$$

Equation (3) postulates that one of the constants,  $u$ ,  $v$ , or  $w$ , has the opposite polarity of the other two. Without restriction we can agree that one of the polarities is negative and the other two are positive.

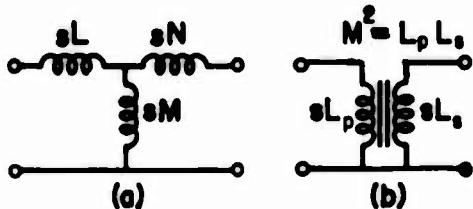
Equation (3) is not a strange one in circuit theory. Assume the particular frequency function  $\phi(s) = s$  and let  $u = L$ ,  $v = M$ , and  $w = N$ , then the two-port

shown in Figure 1 as a block diagram becomes the particular two-port shown in part (a) of Figure 2. It is an inductance star that by

$$1/L + 1/M + 1/N = 0 \quad (4)$$

Figure 2. Inductance Star and Its Equivalent Perfectly Coupled Transformer

has the technical equivalence of a perfectly coupled transformer, as it is shown in part (b) of the figure. The transformer has the primary inductance



$$L_p = L + M, \quad (5a)$$

and the secondary inductance

$$L_s = N + M. \quad (5b)$$

It is customary in transformer theory to identify the product

$$L_p \cdot L_s = M^2 \quad (6a)$$

and the quotient

$$L_p/L_s = n^2 \quad (6b)$$

and to call  $M$  [the same as in Eq. (4)] the mutual inductance and  $n$  the turn ratio of the transformer. It is evident from Eqs. (5a, b) and (6a, b) that the equations

$$L = M(n - 1), \quad (7a)$$

and

$$N = M\left(\frac{1}{n} - 1\right), \quad (7b)$$

satisfy Eq. (4).

Let us now return to Eq. (3). This equation is satisfied when

$$u = v(n - 1), \quad (8a)$$

and

$$w = v\left(\frac{1}{n} - 1\right). \quad (8b)$$

In Eqs. (8a, b)  $n$  is a real constant, not necessarily a positive one. But, when we postulate that only one of the  $u$ ,  $v$ , or  $w$  is negative and the other two are positive, then  $n$  must have the same polarity as  $v$ . When  $v$  is positive,  $u$  or  $w$  is also positive, depending whether  $n$  is greater or smaller than 1 but positive. But when  $v$  is negative,  $u$  and  $w$  are positive only when  $n$  is negative.

Relating to the perfectly coupled transformer let us now define

$$v \text{ as the mutual constant}, \quad (9a)$$

and

$$n \text{ as the ratio constant .} \quad (9b)$$

When  $v$  and  $n$  are negative we will use the notations

$$\bar{v} = -v \quad (10a)$$

and

$$\bar{n} = -n . \quad (10b)$$

We refer to the  $T$  shown in Figure 1 for which Eq. (3) holds, as a perfectly coupled  $T$  and now distinguish three types of perfectly coupled  $T$ 's:

Type (A)	Type (B)	Type (C)
u positive	u negative	u positive
v positive		$v = -\bar{v}$ negative
w negative	w positive	w positive
n positive		$n = -\bar{n}$ negative
$n > 1$	$n < 1$	

### 1.2 The Definition of a Perfectly Coupled and Shunt-Augmented $T$ (pcsa $T$ )

A block diagram of the perfectly coupled and shunt-augmented  $T$ , the main objective of our discussions, is shown in part (a) of Figure 3.

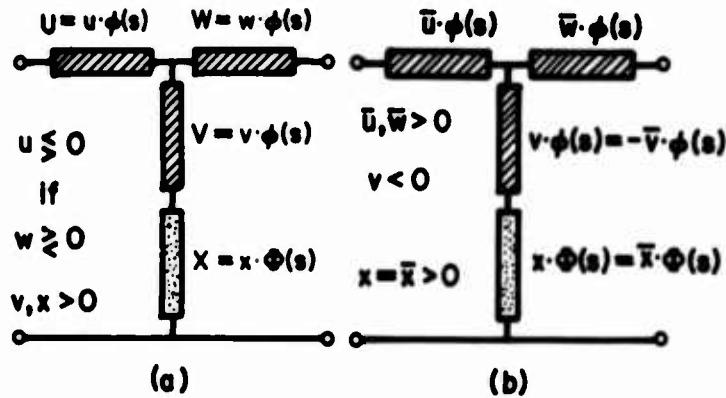


Figure 3. The Perfectly Coupled and Shunt-Augmented  $T$  (pcsa  $T$ ) of Types (A) and (B) in Part (a) and of Type (C) in Part (b).

The upper part of the T is the same as shown in Figure 1. It has the impedance branches  $U = u \cdot \phi(s)$ ,  $V = v \cdot \phi(s)$ , and  $W = w \cdot \phi(s)$ , where  $\phi(s)$  is an arbitrary positive real and normalized frequency function. The constants  $u$ ,  $v$ , and  $w$  have impedance character and are given by Eqs. (8a, b). In the lower part of the T we find the shunt-augmentation  $X$

$$X = x \cdot \Phi(s), \quad (11)$$

which is an impedance.

In Eq. (11)  $x$  is a positive real constant of impedance character and  $\Phi(s)$  denotes a positive real and normalized frequency function with no dimension. In general, we assume that  $\Phi(s)$  is also an arbitrary positive real and normalized frequency function which is in no way related with  $\phi(s)$ . We assume that the perfectly coupled part of the T in part (a) of Figure 3 is of type (A) or (B) so that  $v$  and  $n$  are positive constants.

In part (b) of Figure 3 we show a perfectly coupled and shunt-augmented T where the perfectly coupled part of the T is of type (C). In this event  $\bar{v} = -v$  and  $\bar{n} = -n$  are positive. Merely for formal reasons we used the notation  $\bar{x}$  for the augmentation constant; but, note that this constant is positive. Hence  $x = \bar{x}$ , but  $v = -\bar{v}$ .

We distinguish between three types of pcsa T's:

Type (A)	Type (B)	Type (C)
$u = v(n - 1)$ positive $v$ positive $w = v\left(\frac{1}{n} - 1\right)$ negative	$u = v(n - 1)$ negative $v$ positive $w = v\left(\frac{1}{n} - 1\right)$ positive	$u = \bar{v}(\bar{n} + 1)$ positive $v = -\bar{v}$ negative $w = \bar{v}\left(\frac{1}{\bar{n}} + 1\right)$ positive
$n$ positive $n > 1$	$n < 1$	$n = -\bar{n}$ negative
$x$ positive		$x = \bar{x}$ positive
Coupling frequency function $\phi(s)$	Augmentation frequency function $\Phi(s)$	positive real and normalized functions

According to the preceding table a pcsa T, in addition to being determined by the frequency functions  $\phi(s)$  and  $\Phi(s)$ , can be determined by the three constants

$v$ ,  $x$  and  $n$ , and  $\bar{v}$ ,  $\bar{x}$ , and  $\bar{n}$ , respectively. The branches  $U$  and  $W$  are determined automatically by Eqs. (8a, b).

### 1.3 The Chain Matrix of a pesa T

Any two-port is completely described by the chain matrix

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{1}{E} \begin{vmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{vmatrix}. \quad (12)$$

The chain matrix on the left side of Eq. (12) is based on the two-port equations

$$E_1 = A \cdot E_2 - B \cdot I_2, \quad (13a)$$

$$I_1 = C \cdot E_2 - D \cdot I_2, \quad (13b)$$

where  $E_1$  and  $E_2$  are the port voltages and  $I_1$  and  $I_2$  are the port currents as defined in their positive sense by the arrows in Figure 4. The physical meanings of the ABCD matrix are as follows:

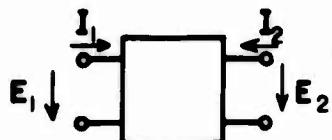


Figure 4. Orientation of the Positive Voltages and Currents at a Passive Two-Port

$$A = \left. \frac{E_1}{E_2} \right|_{I_2=0} \quad (14a)$$

$$B = - \left. \frac{E_1}{I_2} \right|_{E_2=0} \quad (14b)$$

$$C = \left. \frac{I_2}{E_2} \right|_{I_2=0} \quad (14c)$$

$$D = - \left. \frac{I_1}{I_2} \right|_{E_2=0} \quad (14d)$$

According to Eqs. (14a, c) the elements  $A$  and  $C$  can be measured or computed in the status of the open-circuit outport; according to Eqs. (14b, d) the elements  $B$  and  $D$  can be measured or computed in the status of the short-circuit out-port.

In general, the elements of the ABCD matrix are fractions. It is sometimes more convenient to separate a common denominator  $\hat{E}$  from the numerators  $\hat{A}$ ,  $\hat{B}$ ,  $\hat{C}$ , and  $\hat{D}$ . For this reason we prefer to use the matrix that is shown on the right side of Eq. (12).

By the definitions in Eqs. (14a, ..., d) we find, in accordance with Figure 3, part (a):

$$A = \frac{U + V + X}{V + X} , \quad (15a)$$

$$C = \frac{A}{U + V + X} = \frac{1}{V + X} , \quad (15c)$$

$$D = \frac{V + W + X}{V + X} , \quad (15d)$$

$$B = \left( U + \frac{W(V + X)}{V + W + X} \right) D = \frac{UV + UW + VW + (U + W)X}{V + W + X} D \\ = X \frac{U + W}{V + X} . \quad (15b)$$

Note that by Eq. (3)

$UV + UW + VW = 0$  in Eq. (15b). From Eqs. (15a, ..., d) we derive the elements

$$\hat{A} = U + V + X = vn \cdot \phi(s) + x \cdot \Phi(s) , \quad (16a)$$

$$\hat{B} = X(U + W) = vx \frac{(n - 1)^2}{n} \phi(s) \cdot \Phi(s) , \quad (16b)$$

$$\hat{C} = 1 , \quad (16c)$$

$$\hat{D} = V + W + X = \frac{v}{n} \cdot \phi(s) + x \cdot \Phi(s) , \quad (16d)$$

$$\hat{E} = V + X = v \cdot \phi(s) + x \cdot \Phi(s) . \quad (16e)$$

The elements presented in Eqs. (16a, ..., e) describe a pcsa T of type (A) or (B). They are also true for type (C); however, when we prefer to use the notations  $\bar{v}$ ,  $\bar{x}$ , and  $\bar{n}$  for this type, we obtain by the respective replacements:

$$\hat{A} = \bar{v} \bar{n} \cdot \phi(s) + \bar{x} \cdot \Phi(s) , \quad (17a)$$

$$\hat{B} = \bar{v} \bar{x} \frac{(\bar{n} - 1)^2}{\bar{n}} \phi(s) \cdot \Phi(s) , \quad (17b)$$

$$\hat{C} = 1 , \quad (17c)$$

$$\hat{D} = \frac{\bar{v}}{n} \phi(s) + \bar{x} \cdot \Phi(s) ; \quad (17d)$$

$$\hat{E} = -\bar{v} \cdot \phi(s) + \bar{x} \cdot \Phi(s) . \quad (17e)$$

The elements presented in Eqs. (17a, ..., e) describe a pcsa T of type (C).

## 2. A SHUNT TRANSPOSITION BETWEEN THE PORTS OF A pcsa T

Assume a pcsa T of type (A) or (B) that is defined by the constants  $v$ ,  $x$ , and  $n$  besides by the frequency functions  $\phi(s)$  and  $\Phi(s)$ , and assume that this T has an in-port impedance  $X_s = x_s \cdot \Phi(s)$  as shown in part (a) of Figure 7.

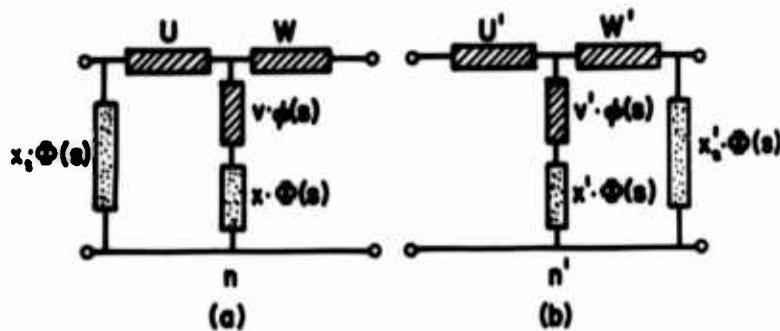


Figure 5. pcsa T of Type (A) or (B) with  
(a) In-Port Impedance and (b) Out-Port  
Impedance

Thus the port impedance and the shunt-augmentation imply the same frequency function  $\Phi(s)$ . The matrix elements of the port impedance are

$$\hat{A}_s = \hat{D}_s = \hat{E}_s = x_s \cdot \Phi(s) , \quad (18a, d, e)$$

$$\hat{B}_s = 0 , \quad (18b)$$

$$\hat{C}_s = 1 . \quad (18c)$$

Remember the rule of matrix multiplication:

$$\frac{1}{\hat{E}_1} \begin{vmatrix} \hat{A}_1 & \hat{B}_1 \\ \hat{C}_1 & \hat{D}_1 \end{vmatrix} \frac{1}{\hat{E}_2} \begin{vmatrix} \hat{A}_2 & \hat{B}_2 \\ \hat{C}_2 & \hat{D}_2 \end{vmatrix} = \frac{1}{\hat{E}_1 \hat{E}_2} \begin{vmatrix} \hat{A}_1 \hat{A}_2 + \hat{B}_1 \hat{C}_2 & \hat{A}_1 \hat{B}_2 + \hat{B}_1 \hat{D}_2 \\ \hat{C}_1 \hat{A}_2 + \hat{D}_1 \hat{C}_2 & \hat{C}_1 \hat{B}_2 + \hat{D}_1 \hat{D}_2 \end{vmatrix} \quad (19)$$

Since the matrix of a tandem results from the product of its two-ports, we obtain the elements of the pcsa T with its in-port impedance as

$$\hat{A} = \left| v n x_s \cdot \phi(s) + x x_s \cdot \Phi(s) \right| \Phi(s) , \quad (20a)$$

$$\hat{B} = v x x_s \frac{(n-1)^2}{n} \phi(s) \cdot \Phi^2(s) , \quad (20b)$$

$$\hat{C} = v n \cdot \phi(s) + (x + x_s) \Phi(s) , \quad (20c)$$

$$\hat{D} = \left| \frac{v}{n} (x(n-1)^2 + x_s) \phi(s) + x x_s \cdot \Phi(s) \right| \Phi(s) , \quad (20d)$$

$$\hat{E} = \left| v \cdot \phi(s) + x \cdot \Phi(s) \right| x_s \cdot \Phi(s) . \quad (20e)$$

The elements in Eqs. (20a, ..., e) are obtained by the rule, Eq. (19), where the index 1 refers to the port impedance given by its elements in Eqs. (18a, ..., e), and where the index 2 refers to the pcsa T given by its elements in Eqs. (10a, ..., e).

Consider now the structure in part (b) of Figure 7. The pcsa T of type (A) or (B) in this part is defined by the constants  $v'$ ,  $x'$ , and  $n'$  besides by the frequency functions  $\phi(s)$  and  $\Phi(s)$ . The T has an out-port impedance  $X'_s = x'_s \cdot \Phi(s)$ . By the same rule used previously we find the elements of the two-port in part (b) of Figure 7 as

$$\hat{A}' = \left[ v' \left( n' x'_s + x' - \frac{(n'-1)^2}{n'} \right) \phi(s) + x' x'_s \cdot \Phi(s) \right] \Phi(s) , \quad (21a)$$

$$\hat{B}' = v' x' x'_s \frac{(n'-1)^2}{n'} \phi(s) \cdot \Phi(s) , \quad (21b)$$

$$\hat{C}' = \frac{v'}{n'} \phi(s) + (x' + x'_s) \Phi(s) , \quad (21c)$$

$$\hat{D}' = \left[ \frac{v'}{n'} \phi(s) + x' \cdot \Phi(s) \right] x'_s \cdot \Phi(s) , \quad (21d)$$

$$\hat{E}' = \left[ v' \cdot \phi(s) + x' \cdot \Phi(s) \right] x'_s \cdot \Phi(s) . \quad (21e)$$

Equations (21a, ..., e) are obtained by applying the rule, Eq. (19), so that the index 1 refers to the elements of the pcsa T given in Eqs. (16a, ..., e) and index 2 refers to the elements of the out-port impedance given in Eqs. (18a, ..., e).

We will now show that both two-ports in Figure 5 can become equivalent.

Two-ports are defined as equivalent if they have the same chain matrix. That means that both chain matrices have the same elements  $\hat{A}, \dots, \hat{E}$ .

The two-ports in Figure 5 are equivalent when we can ascertain that the elements in Eqs. (20a, ..., e) are the same as those in Eqs. (21a, ..., e). Note that a matrix remains unchanged if we multiply all numerator elements  $\hat{A}, \dots, \hat{D}$  and the denominator element  $\hat{E}$  by the same factor. We start the enforcement of the element identities conveniently with the denominator elements  $\hat{E}$  and  $\hat{E}'$ . Suppose we divide all elements in Eqs. (20a, ..., e) by  $v x_s \cdot \Phi(s)$ , and we divide all elements in Eqs. (21a, ..., e) by  $v' x'_s \cdot \Phi(s)$ . Then we obtain

$$\hat{E} = \phi(s) + \frac{x}{v} \Phi(s), \quad (22e)$$

and

$$\hat{E}' = \phi(s) + \frac{x'}{v'} \Phi(s). \quad (23e)$$

If  $\hat{E} = \hat{E}'$ , it is necessary that

$$\frac{v}{x} = \frac{v'}{x'} = K. \quad (24)$$

Since  $v$  and  $x$ , and  $v'$  and  $x'$  are defined as positive constants [we suppose that the pcsa T's are of types (A) or (B)], the notation  $K$  in Eq. (24) is a positive constant. Dividing the numerator elements in Eqs. (20a, ..., d) by  $v x_s \cdot \Phi(s)$  and the numerator elements in Eqs. (21a, ..., d) by  $v' x'_s \cdot \Phi(s)$  we obtain:

$$\hat{A} = n \cdot \phi(u) + \frac{x}{v} \Phi(s), \quad (22a)$$

$$\hat{B} = x \frac{(n-1)^2}{n} \phi(s) \cdot \Phi(s), \quad (22b)$$

$$\hat{C} = \frac{n}{x_s} \frac{\phi(s)}{\Phi(s)} + \frac{x+x_s}{v x_s}, \quad (22c)$$

$$\hat{D} = \frac{1}{n} \left[ \frac{x}{x_s} (n-1)^2 + 1 \right] \phi(s) + \frac{x}{v} \Phi(s), \quad (22d)$$

and

$$\hat{A}' = \left[ n' + \frac{x'}{x'_s} \frac{(n' - 1)^2}{n'} \right] \phi(s) + \frac{x'}{v'} \Phi(s) , \quad (23a)$$

$$\hat{B}' = x' \frac{(n' - 1)^2}{n'} \phi(s) \cdot \Phi(s) , \quad (23b)$$

$$\hat{C}' = \frac{1}{x'_s n'} \frac{\phi(s)}{\Phi(s)} + \frac{x' + x'_s}{v' x'_s} , \quad (23c)$$

$$\hat{D}' = \frac{1}{n'} \phi(s) + \frac{x'}{v'} \Phi(s) . \quad (23d)$$

In order to make  $A = A'$  it is necessary that

$$n = n' + \frac{x'}{x'_s} \frac{(n' - 1)^2}{n'} . \quad (25)$$

The identity  $x/v = x'/v'$  is already given by Eq. (24).

In order to make  $C = C'$  it is necessary that

$$\frac{n}{x'_s} = \frac{1}{x'_s n'} , \quad (26)$$

and that

$$\frac{x + x'_s}{v x'_s} = \frac{x' + x'_s}{v' x'_s} . \quad (27)$$

Equation (25) expresses  $n$  by exclusively primed constants  $x'$ ,  $x'_s$ , and  $n'$ . With  $n$  known by this equation we obtain using Eq. (26)

$$x'_s = x'_s \cdot n n' . \quad (28)$$

Substituting  $v/x = K$  and  $v = xK$  into Eq. (27) we obtain

$$\frac{1}{x} = \frac{1}{x'} + \frac{1}{x'_s} - \frac{1}{x'_s} . \quad (29)$$

Finally by Eq. (24)

$$v' = x' \cdot K . \quad (30)$$

Hence, if the two-ports in Figure 5 are equivalent, we are now able to compute the constants of the two-port in part (a) when the constants of the two-port in part (b) are known; we need only to apply Eqs. (25), (28), (29), and (30) in sequence. But the proof of the equivalence is not yet finished; we still have to compare  $\hat{B}$  and  $\hat{B}'$ , and  $\hat{D}$  and  $\hat{D}'$ . In order to make  $\hat{B} = \hat{B}'$ , it is necessary that

$$x' \frac{(n'-1)^2}{n'} = x \frac{(n-1)^2}{n} . \quad (31)$$

In order to make  $\hat{D} = \hat{D}'$ , it is necessary that

$$n' = n \frac{x_s}{x_s + x(n-1)^2} . \quad (32)$$

Like Eq. (25), Eq. (32) presents  $n'$  exclusively by the constants of the two-port in part (a) of Figure 5. Knowing  $n'$ , we are able to compute by Eq. (26)

$$x'_s = \frac{x_s}{n n'} . \quad (33)$$

Knowing  $x'_s$  by Eq. (33) we obtain by Eq. (29)

$$\frac{1}{x'} = \frac{1}{x} + \frac{1}{x_s} - \frac{1}{x'_s} . \quad (34)$$

Finally by Eq. (24)

$$v' = x' \cdot K . \quad (35)$$

Hence, if the two-ports in Figure 5 are equivalent, we are able to compute the constants of the two-port in part (b) when the constants of the two-port in part (a) are known; we need only to apply Eqs. (32), (33) and (34) in sequence. These equations are the inverse of Eqs. (25), (28), (29), and (30).

We will now prove that Eq. (31) is true. By this equation

$$\frac{x}{x'} = \frac{n}{n'} - \frac{(n' - 1)^2}{(n - 1)^2} \quad . \quad (36)$$

By Eq. (32)

$$\frac{n}{n'} = 1 + \frac{x}{x_s} (n - 1)^2 \quad , \quad (37a)$$

and

$$(n' - 1)^2 = (n - 1)^2 \left[ \frac{1 - x(n - 1)/x_s}{1 + x(n - 1)^2/x_s} \right]^2 \quad . \quad (37b)$$

Then

$$\frac{x}{x'} = \frac{\left[ 1 - x(n - 1)/x_s \right]^2}{1 + x(n - 1)^2/x_s} \quad . \quad (38)$$

On the other hand, by Eq. (34)

$$\begin{aligned} \frac{x}{x'} &= 1 + \frac{x}{x_s} - \frac{x}{x_s} n n' \\ &= 1 + \frac{x}{x_s} \left[ 1 - \frac{n^2}{1 + x(n - 1)^2/x_s} \right] \\ &= \frac{\left[ 1 - x(n - 1)/x_s \right]^2}{1 + x(n - 1)^2/x_s} \quad , \end{aligned}$$

which is the same result obtained in Eq. (38).

Thus we proved that if either the set of Eqs. (25), (28), (29), and (30), or the set of inverse Eqs. (32), (33), (34) and (35) is true then the two-ports shown in parts (a) and (b) of Figure 5, are equivalent.

The two-ports in Figure 5 may also be of type (C). The equivalence also holds for the following sets of equations:

$$\bar{n} = \bar{n}' + \frac{\bar{x}'_s}{\bar{x}_s} \frac{(\bar{n}' + 1)^2}{\bar{n}'} , \quad (39)$$

$$x_s = x'_s \cdot \bar{n} \bar{n}' , \quad (40)$$

$$\frac{1}{x} = \frac{1}{\bar{x}'} + \frac{1}{x'_s} - \frac{1}{x_s} , \quad (41)$$

$$\bar{v} = \bar{x} \cdot K , \quad (42)$$

and

$$\bar{n}' = \bar{n} \frac{x_s}{x_s + \bar{x} (\bar{n} + 1)^2} , \quad (43)$$

$$x'_s = \frac{x_s}{\bar{n} \bar{n}'} , \quad (44)$$

$$\frac{1}{x'} = \frac{1}{\bar{x}} + \frac{1}{x_s} - \frac{1}{x'_s} , \quad (45)$$

$$\bar{v}' = \bar{x}' \cdot K . \quad (46)$$

The equivalence is shown in Figure 6 where for the pcsa T of type (C) the constants marked by a bar are used.

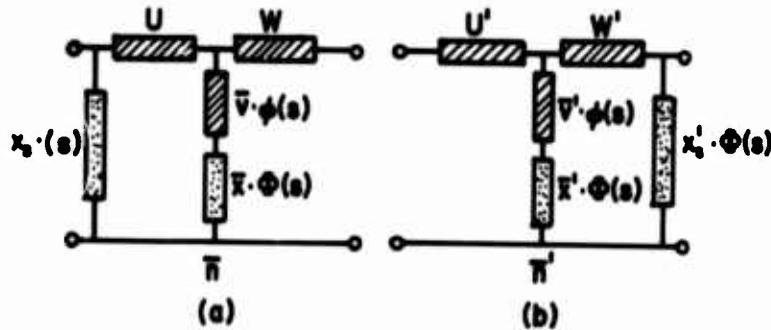


Figure 6. pcsa T of Type (C) with (a) In-Port Impedance and (b) Out-Port Impedance

We are now able to state the following theorem:

THEOREM 1 (concerning the transposition of port impedances)

A pcsa  $T$  of type (A) or (B) that has an impedance  $X_s = x_s \cdot \Phi(s)$  at its in-port is equivalent to a pcsa of the same type that has an impedance  $X'_s = x'_s \cdot \Phi(s)$  at its out-port when the port impedances imply the same frequency function  $\Phi(s)$  as the shunt-augmentation and when the set of Eqs. (25), (28) (29), and (30) or of Eqs. (32), (33), (34) and (35) holds. Also, two pcsa  $T$ 's of type (C), each having port impedance  $X_s$  and  $X'_s$ , respectively, are equivalent if the port impedances imply the same frequency function as the augmentations, and when either the set of Eqs. (39), ..., (42) or the set of Eqs. (43), ..., (46) holds.

The equivalence between the two-ports in Figures 5 and 6 can also be interpreted in such a way that it looks like the respective port impedance has been transposed to the other port, whereby unprimed constants change to primed ones and vice versa according to the respective set of the aforementioned equations. Hence, we can also state the following theorem:

THEOREM 2 (concerning the transposition of port impedances)

When a pcsa  $T$  has an in-port impedance  $X_s$  that implies the same frequency function  $\Phi(s)$  as the shunt-augmentation, then the port impedance can be transposed to the out-port where it appears as  $X'_s$  and vice versa. By the transposition, unprimed constants are changed to primed ones and vice versa according to the sets of equations, Eqs. (25), (28), (29), (30) or Eqs. (32), (33), (34), (35), if the  $T$  is of type (A) or (B). If it is of type (C), the constants change according to the sets of Eqs. (39), (40), (42) or Eqs. (43), (44), (45), (46).

Note that by the transposition the constant  $K$  given in Eq. (24) remains unchanged. It is essential to recognize that by the transposition we are able to change the ratio constants  $n$  and  $n'$ , or  $\bar{n}$  and  $\bar{n}'$ . If we transpose in the forward direction, Eq. (32) shows that this ratio decreases from  $n$  to  $n'$  if the pcsa  $T$  is of type (A) or of type (B). It is therefore possible that a type (A)  $T$  becomes a type (B)  $T$  after the transposition. Equation (25), which is the inverse of Eq. (32), shows that by a transposition in the backward direction the ratio constant increases from  $n'$  to  $n$  if the  $T$  is of type (A) or of type (B). Similarly, we recognize by

Eq. (43) that by a forward transposition the magnitude of the ratio constant also decreases from  $\bar{n}$  to  $\bar{n}'$ , whereas it increases from  $\bar{n}'$  to  $\bar{n}$  according to Eq. (39) by a backward transposition if in both events the T is of type (C).

### 3. MATCHED AND PERFECTLY MATCHED TANDEMS OF TWO pcsa T's

#### 3.1 Definitions

In Figure 7 we show a tandem consisting of two pcsa T sections, the left one being of type (A) or of type (B), the right one being of type (C).

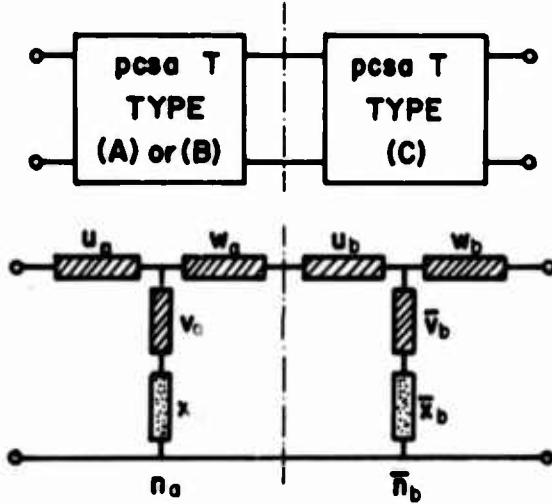


Figure 7. Tandem of Type (AC) or (BC)

of type (C) are defined as positive, the notation  $K$  in Eq. (47) is necessarily a positive constant, too. Equation (47) also allows us to introduce another constant  $k_a$  that is defined as

$$k_a = \frac{v_a}{v_b} = \frac{x_a}{x_b} . \quad (48)$$

#### 3.1.1 DEFINITION OF A MATCHED TANDEM

The pcsa T sections in a matched tandem are such that the sections imply the same frequency functions  $\phi(s)$  and  $\Phi(s)$  and that the constants  $v, x$  and  $\bar{v}, \bar{x}$  of the shunt impedances of the T's are proportional, as shown by Eq. (47).

Let us assume that the two sections in Figure 7 are not completely arbitrary. Reasonably we presume at least that both sections imply the same frequency function  $\phi(s)$  (marked by diagonal shading) and the same frequency function  $\Phi(s)$  (marked by dot shading). In addition to this we presume that

$$\frac{v_a}{x_a} = \frac{\bar{v}_b}{\bar{x}_b} = K . \quad (47)$$

Since the constants  $v$  and  $x$  in a pcsa T of types (A) or (B) and the constants  $\bar{v} = -v$  and  $\bar{x} = x$  in a pcsa T

We will now develop the elements  $\hat{A}, \dots, \hat{E}$  of the chain matrix of the tandem shown in Figure 7.

By Eqs. (47) and (48) we can reduce the number of notations:

By Eq. (47)

$$x_a = \frac{v_a}{K}, \quad (49)$$

by Eq. (48)

$$\bar{v}_b = \frac{v_a}{k_a}, \quad (50)$$

$$\bar{x}_b = \frac{v_a}{K k_a}. \quad (51)$$

The matrix elements of the section shown at the left in Figure 7 are obtained by Eqs. (16a, ..., e) as

$$\hat{A}_a = n_a \cdot \phi(s) + \frac{1}{K} \cdot \Phi(s), \quad (52a)$$

$$\hat{B}_a = \frac{v_a}{K} \cdot \frac{(n_a - 1)^2}{n_a} \phi(s) \cdot \Phi(s), \quad (52b)$$

$$\hat{C}_a = \frac{1}{v_a}, \quad (52c)$$

$$\hat{D}_a = \frac{1}{n_a} \phi(s) + \frac{1}{K} \Phi(s), \quad (52d)$$

$$\hat{E}_a = \frac{1}{K} \Phi(s) + \phi(s). \quad (52e)$$

The elements of the matrix of the section shown at the right in Figure 7 are obtained by Eqs. (17a, ..., e) as

$$\hat{A}_b = \bar{n}_b \cdot \phi(s) + \frac{1}{K} \Phi(s), \quad (53a)$$

$$\hat{B}_b = \frac{v_a}{K k_a} \frac{(\bar{n}_b + 1)^2}{\bar{n}_b} \phi(s) \cdot \Phi(s) , \quad (53b)$$

$$\hat{C}_b = \frac{k_a}{v_a} , \quad (53c)$$

$$\hat{D}_b = \frac{1}{\bar{n}_b} \phi(s) + \frac{1}{K} \Phi(s) , \quad (53d)$$

$$\hat{E}_b = \frac{1}{K} \Phi(s) - \phi(s) . \quad (53e)$$

By the multiplication rule of matrices that is given in Eq. (19) we obtain the matrix elements of the chain matrix of the tandem in Figure 7. In order to distinguish the result from a later one we use the subindex (AB)C. This index shows that in Figure 7 either a section of type (A) or of type (B) is followed in the forward direction by a section of type (C).

$$\hat{A}_{(AB)C} = n_a \bar{n}_b \cdot \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{K} \left[ (n_a + \bar{n}_b) + \frac{(n_a - 1)^2}{n_a} k_a \right] \phi(s) \cdot \Phi(s) , \quad (54a)$$

$$\begin{aligned} \hat{B}_{(AB)C} = & \frac{v_a}{K} \left[ \left( n_a \frac{(\bar{n}_b + 1)^2}{\bar{n}_b k_a} + \frac{(n_a - 1)^2}{n_a \bar{n}_b} \right) \phi(s) \right. \\ & \left. + \frac{1}{K} \left( \frac{(\bar{n}_b + 1)^2}{k_a \bar{n}_b} + \frac{(n_a - 1)^2}{n_a} \right) \Phi(s) \right] \phi(s) \cdot \Phi(s) , \end{aligned} \quad (54b)$$

$$\hat{C}_{(AB)C} = \frac{1}{v_a} \left[ \left( \bar{n}_b + \frac{k_a}{n_a} \right) \phi(s) + \frac{1}{K} (1 + k_a) \Phi(s) \right] , \quad (54c)$$

$$\hat{D}_{(AB)C} = \frac{1}{n_a \bar{n}_b} \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{K} \left[ \frac{1}{n_a} + \frac{1}{\bar{n}_b} + \frac{(\bar{n}_b + 1)^2}{\bar{n}_b k_a} \right] \phi(s) \cdot \Phi(s) , \quad (54d)$$

$$\hat{E}_{(AB)C} = \frac{1}{K^2} \Phi^2(s) - \phi^2(s) . \quad (54e)$$

Let us now impose a stronger restrain on the matching of the T's. Let us demand that the ratio constants  $n_a$  and  $\bar{n}_b$  be inverse each other, given by

$$n_a = \frac{1}{\bar{n}_b} = n_0 . \quad (55)$$

The constant  $n_0$  is positive and greater than 1 if the left side section in Figure 7 is of type (A), and it is smaller than 1 if it is of type (B).

### 3.1.2 DEFINITION OF THE PERFECTLY MATCHED TANDEM

If in a matched tandem the magnitudes of the ratio constants of the sections are such that they are vice versa inverse, we refer to such a tandem as being perfectly matched.

#### 3.2 Equivalent Perfectly Matched Tandems

Assume that the left side section in Figure 7 is of type (A) and assume that Eq. (55) with  $n_0 > 1$  holds. Then the matrix elements of this tandem are:

$$\hat{A}_{AC} = \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{Kn_0} \left[ n_0^2 + 1 + (n_0 - 1)^2 k_a \right] \phi(s) \cdot \Phi(s) , \quad (56a)$$

$$\hat{B}_{AC} = \frac{v_a}{K k_a} \left( (n_0 + 1)^2 + k_a (n_0 - 1)^2 \right) \left[ \phi(s) + \frac{1}{Kn_0} \Phi(s) \right] \phi(s) \cdot \Phi(s) , \quad (56b)$$

$$\hat{C}_{AC} = \frac{1 + k_a}{v_a} \left[ \frac{1}{n_0} \phi(s) + \frac{1}{K} \Phi(s) \right] , \quad (56c)$$

$$\hat{D}_{AC} = \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{Kn_0} \left[ n_0^2 + 1 + \frac{(n_0 + 1)^2}{k_a} \right] \phi(s) \cdot \Phi(s) , \quad (56d)$$

$$\hat{E}_{AC} = \frac{1}{K^2} \Phi^2(s) - \phi^2(s) . \quad (56e)$$

When the left side section in Figure 7 is of type (B), the matrix elements  $\hat{A}_{BC}, \dots, \hat{E}_{BC}$  are the same expressions as given in Eqs. (56a, ..., e) with the only exception being that in this event  $n_0 < 1$ .

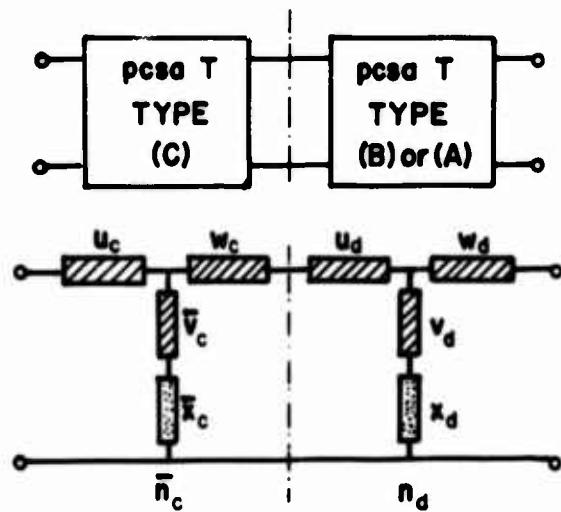


Figure 8. Tandem of Type (CB) or (CA)

$$\frac{\bar{v}_c}{\bar{x}_c} = \frac{v_d}{x_d} = K. \quad (57)$$

and

$$\frac{v_d}{v_c} = \frac{x_d}{\bar{x}_c} = k_b. \quad (58)$$

We now relate the two tandems shown in Figures 7 and 8 by assuming that the constant  $K$  in Eq. (57) is of the same value as the constant  $K$  in Eq. (47). We do not postulate that the constants  $k_a$  and  $k_b$  in Eqs. (48) and (58) to be the same; however, it should be noted that the constants  $k_a$  and  $k_b$ , which express ratios between the mutual impedance constants  $v$  and  $\bar{v}$  and the augmentation constants  $x$  and  $\bar{x}$ , are defined such that the unbarred notations are in the numerators and the barred notations are in the denominators.

Again let us reduce the notations by using  $v_d$  only; we set

$$x_d = \frac{v_d}{K}, \quad (59)$$

$$\frac{\bar{v}_c}{\bar{x}_c} = \frac{v_d}{k_b}, \quad (60)$$

Consider now the tandem shown in Figure 8. In this tandem, inverse in the section sequence to Figure 7, a pcsa T of type (C) is followed in the forward direction by another section that is either of type (B) or of type (A). We will relate the tandems in Figures 7 and 8 somewhat later; hence, we assume that the right-side section in Figure 8 is of type (B) if the left-side section in Figure 7 is of type (A) and vice versa. We assume that the sections in Figure 8 are matched, too, so that

$$\bar{x}_c = \frac{v_d}{K k_b} . \quad (61)$$

Applying Eqs. (17a, ..., e) we obtain the matrix elements of the left side section in Figure 8 as

$$\hat{A}_c = \bar{n}_c \cdot \phi(s) + \frac{1}{K} \Phi(s) , \quad (62a)$$

$$\hat{B}_c = \frac{v_d}{K k_b} \frac{(\bar{n}_c + 1)^2}{\bar{n}_c} \phi(s) \cdot \Phi(s) , \quad (62b)$$

$$\hat{C}_c = \frac{k_b}{v_d} , \quad (62c)$$

$$\hat{D}_c = \frac{1}{\bar{n}_c} \phi(s) + \frac{1}{K} \Phi(s) , \quad (62d)$$

$$\hat{E}_c = \frac{1}{K} \Phi(s) - \phi(s) . \quad (62e)$$

Applying Eqs. (16a, ..., e) we obtain the matrix elements of the right side section in Figure 8 as

$$\hat{A}_d = n_d \cdot \phi(s) + \frac{1}{K} \Phi(s) , \quad (63a)$$

$$\hat{B}_d = \frac{v_d}{K} \frac{(n_d - 1)^2}{n_d} \phi(s) \cdot \Phi(s) , \quad (63b)$$

$$\hat{C}_d = \frac{1}{v_d} , \quad (63c)$$

$$\hat{D}_d = \frac{1}{n_d} \phi(s) + \frac{1}{K} \Phi(s) , \quad (63d)$$

$$\hat{E}_d = \frac{1}{K} \Phi(s) + \phi(s) . \quad (63e)$$

By matrix multiplication we obtain the elements of the tandem in Figure 8 as

$$\hat{A}_{C(BA)} = \bar{n}_c n_d \cdot \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{K} \left[ \bar{n}_c + n_d + \frac{(\bar{n}_c + 1)^2}{k_b \bar{n}_c} \right] \phi(s) \cdot \Phi(s) , \quad (64a)$$

$$\hat{B}_{C(BA)} = \frac{v_d}{K} \left[ \left( \frac{(\bar{n}_c - 1)^2}{n_d} + \frac{(\bar{n}_c + 1)^2}{k_b \bar{n}_c n_d} \right) \phi(s) + \frac{1}{K} \left( \frac{(\bar{n}_c - 1)^2}{n_d} + \frac{(\bar{n}_c + 1)^2}{k_b \bar{n}_c} \right) \Phi(s) \right] \phi(s) \cdot \Phi(s) , \quad (64b)$$

$$\hat{C}_{C(BA)} = \frac{1}{v_d} \left[ \left( \frac{1}{\bar{n}_c} + n_d k_b \right) \phi(s) + \frac{1}{K} (1 + k_b) \Phi(s) \right] , \quad (64c)$$

$$\hat{D}_{C(BA)} = \frac{1}{\bar{n}_c n_d} \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{K} \left[ \frac{1}{\bar{n}_c} + \frac{1}{n_d} + \frac{(\bar{n}_c - 1)^2}{n_d} \right] \phi(s) \cdot \Phi(s) , \quad (64d)$$

$$\hat{E}_{C(BA)} = \frac{1}{K^2} \Phi^2(s) - \phi^2(s) . \quad (64e)$$

Let us now assume that the tandem in Figure 8 is also perfectly matched by setting

$$\bar{n}_c = \frac{1}{n_d} = n_0 , \quad (65)$$

and let us assume that  $n_0$  is of the same value in Eqs. (65) and (55). Thus, when  $n_0 > 1$ , the right side section in Figure 8 is of type (B). Substituting  $n_0$  in Eqs. (64a, . . . , e) we obtain the following matrix elements:

$$\hat{A}_{CB} = \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{K n_0} \left[ n_0^2 + 1 + \frac{(n_0 + 1)^2}{k_b} \right] \phi(s) \cdot \Phi(s) , \quad (66a)$$

$$\hat{B}_{CB} = \frac{v_d}{K} \left[ (n_0 - 1)^2 + \frac{(n_0 + 1)^2}{k_b} \right] \left( \phi(s) + \frac{1}{K n_0} \Phi(s) \right) \phi(s) \cdot \Phi(s) , \quad (66b)$$

$$\hat{C}_{CB} = \frac{1 + k_b}{v_d} \left[ \frac{1}{n_0} \phi(s) + \frac{1}{K} \Phi(s) \right] , \quad (66c)$$

$$\hat{D}_{CB} = \phi^2(s) + \frac{1}{K^2} \Phi^2(s) + \frac{1}{K n_0} \left[ n_0^2 + 1 + (n_0 - 1)^2 k_b \right] \phi(s) \cdot \Phi(s) , \quad (66d)$$

$$\hat{E}_{CB} = \frac{1}{K^2} \Phi^2(s) - \phi^2(s) . \quad (66e)$$

In Eqs. (66a, . . . , e) we assume that  $n_0$  is greater than 1. If it is smaller than 1, then the right side section in Figure 8 is of type (A) and the respective elements of the matrix  $\hat{A}_{CA}, \dots, \hat{E}_{CA}$  are given by the same Eqs. (66a, . . . , e) with the only exception being that  $n_0 < 1$ .

Similarities in the system of Eqs. (66a, . . . , e) and of Eqs. (56a, . . . , e) suggest investigating whether it is possible that a perfectly matched tandem shown in Figure 8 is equivalent to a tandem shown in Figure 7 when Eqs. (65), (55) and Eqs. (47), (57) are true. The tandems are equivalent when their chain matrices are the same in all their elements. We will now find the conditions under which equivalence can be obtained.

Equations (56e) and (663) are completely identical.

Equations (56a) and (66a) become identical if

$$k_a k_b = \left[ \frac{n_0 + 1}{n_0 - 1} \right]^2 . \quad (67)$$

Under the same condition, Eq. (67) and Eq. (56d) become identical with Eq. (66d).

Equations (56b) and (66b) become identical if

$$\frac{v_a}{v_d} = \frac{k_a (n_0 - 1)^2 + k_a (n_0 + 1)^2 / k_b}{k_a (n_0 - 1)^2 + (n_0 + 1)^2} ,$$

or by Eq. (67)

$$\frac{v_a}{v_d} = \frac{1 + k_a}{1 + k_b} . \quad (68)$$

Under the same condition Eq. (68) and Eq. (56c) become identical with Eq. (66c).

Let us now draw the following conclusion: let the tandem shown in Figure 7 be a perfectly matched one by Eq. (55); let the tandem shown in Figure 8 also be perfectly matched by Eq. (65). Then

$$n_a = \frac{1}{n_b} = \bar{n}_c = \frac{1}{n_d} = n_0 . \quad (69)$$

If  $n_0 > 1$ , then the left side pcra T section in Figure 7 is of type (A) and the right side pcra T section in Figure 8 is of type (B). Types (A) and (B) are interchanged if  $n_0 < 1$ .

By Eqs. (47) and (57)

$$K = \frac{v_a}{x_a} = \frac{\bar{x}_b}{\bar{x}_b} = \frac{\bar{x}_c}{\bar{x}_c} = \frac{v_d}{x_d} . \quad (70)$$

We defined  $k_a$  by Eq. (48) and  $k_b$  by Eq. (58).

The tandems shown in Figures 7 and 8 are equivalent

- (1) if Eq. (70) holds;
- (2) if their sections are perfectly matched and Eq. (69) holds;
- (3) if Eqs. (67) and (68) hold.

The equivalence allows us to determine the perfectly matched tandem Figure 8 when the tandem Figure 7 is known. By Eqs. (47), (48), and (55) we know  $K$ ,  $k_a$ , and  $n_0$ . Then by Eq. (67) we obtain  $k_b$ , and with this by Eq. (68) we obtain  $v_d$ . Eq. (57) now gives us  $x_d = v_d/K$ , and Eq. (58) gives us  $\bar{x}_c = x_d/k_b$  and  $\bar{v}_c = v_d/k_b$ . Finally by Eq. (65) the ratio constants  $\bar{n}_c$  and  $1/n_d$  are known and thus the complete tandem in Figure 8 is known.

We are now able to state the following theorem:

**THEOREM 3 (concerning the equivalence of perfectly matched tandems)**

A tandem in which a pcsa T of type (A) or of type (B) is followed by a pcsa T of type (C) has an equivalent tandem in which a pcsa T of type (C) is followed by a pcsa T of type (B) or of type (A) if in each tandem the pcsa T sections are perfectly matched and Eqs. (67) and (68) hold.

**4. SYMMETRICAL PERFECTLY MATCHED TANDEMS**

A two-port is defined as being symmetrical if the elements A and D of the chain matrix are the same. We can easily recognize that if

$$k_a = \left| \frac{n_0 + 1}{n_0 - 1} \right|, \quad (71)$$

then  $\hat{A}_{AC}$  in Eq. (56a) becomes identical with  $\hat{D}_{AC}$  in Eq. (56d). Similarly, if

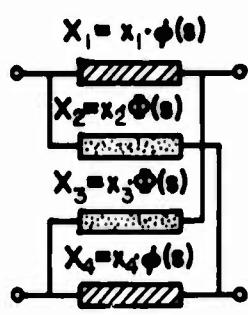
$$k_b = \left| \frac{n_0 + 1}{n_0 - 1} \right|, \quad (72)$$

then  $\hat{A}_{CB}$  in Eq. (66a) becomes  $\hat{D}_{CB}$  in Eq. (66d). In both events it is assumed that  $n_0 > 1$ . If  $n_0 < 1$ , the same statements are true. Then the element  $\hat{A}_{BC}$  becomes identical with the element  $\hat{D}_{BC}$  by Eq. (71) and the element  $\hat{A}_{CA}$  becomes identical with the element  $\hat{D}_{CA}$  by Eq. (72).

**5. A PERFECTLY MATCHED TANDEM AND ITS EQUIVALENT LATTICE TWO-PORT**

Consider the lattice two-port shown in Figure 9. In a lattice we can evidently distinguish two pairs of branches: one pair of branches in Figure 9 has the numerical indices 1 and 4, the other pair has the indices 2 and 3. Let the branch notations in this

figure be impedances. As indicated by the diagonal shading, the branches



$$X_1 = x_1 \cdot \phi(s), \text{ and} \quad (73a)$$

$$X_4 = x_4 \cdot \phi(s) \quad (73b)$$

Figure 9. Lattice Two-Port

imply the normalized frequency function  $\phi(s)$  that is also implied in the perfectly coupled branches of the pcsa  $T$ . Dot shading indicates that the branches

$$X_2 = x_2 \cdot \Phi(s) , \text{ and} \quad (74a)$$

$$X_3 = x_3 \cdot \Phi(s) \quad (74b)$$

imply the normalized frequency function  $\Phi(s)$  that is also implied in the augmentation of the pcsa  $T$ . Since  $X_1, \dots, X_4$  are positive impedances, the  $x_{1,4}$  and  $x_{2,3}$  in Eqs. (73a,b) and (74a,b) are positive real constants of impedance character.

It is well known that a lattice has the following elements of its chain matrix:

$$\hat{A}_X = (X_1 + X_3)(X_2 + X_4) , \quad (75a)$$

$$\hat{B}_X = X_1 X_4 (X_2 + X_3) + X_2 X_3 (X_1 + X_4) , \quad (75b)$$

$$\hat{C}_X = X_1 + X_2 + X_3 + X_4 , \quad (75c)$$

$$\hat{D}_X = (X_1 + X_2)(X_3 + X_4) , \quad (75d)$$

$$\hat{E}_X = X_2 X_3 - X_1 X_4 . \quad (75e)$$

Let us now substitute the explicit expressions given in Eqs. (73a,b) and (74a,b) and let us divide all the elements by the product  $x_1 x_4$ . We then obtain:

$$\hat{A}_X = \phi^2(s) + \frac{x_2 x_3}{x_1 x_4} \Phi^2(s) + \frac{x_1 x_2 + x_3 x_4}{x_1 x_4} \phi(s) \cdot \Phi(s) , \quad (76a)$$

$$\hat{B}_X = (x_2 + x_3) \left[ \phi(s) + \frac{x_2 x_3}{x_1 x_4} \frac{x_1 + x_4}{x_2 + x_3} \Phi(s) \right] \phi(s) \cdot \Phi(s) , \quad (76b)$$

$$\hat{C}_X = \frac{(x_1 + x_4) \phi(s) + (x_2 + x_3) \Phi(s)}{x_1 x_4} , \quad (76c)$$

$$\hat{D}_X = \phi^2(s) + \frac{x_2 x_3}{x_1 x_4} \Phi^2(s) + \frac{x_1 x_3 + x_2 x_4}{x_1 x_4} \phi(s) \cdot \Phi(s) , \quad (76d)$$

$$\hat{E}_X = \frac{x_2 x_3}{x_1 x_4} \Phi^2(s) - \phi^2(s) . \quad (76e)$$

There are similarities which suggest investigating whether the elements presented in Eqs. (76a, ..., e) can become identical with the elements presented in Eqs. (56a, ..., e) and in Eqs. (66a, ..., e). This means that there is a possibility that a perfectly matched tandem of two pcesa T's as discussed in Section 4 is equivalent to the lattice shown in Figure 9. Let us first find the identities between the elements in Eqs. (76a, ..., e) and Eqs. (56a, ..., e).

Equations (76e) and (56e) become identical if

$$\frac{x_1 x_4}{x_2 x_3} = K^2 . \quad (77)$$

Thus

$$K = \pm \sqrt{\frac{x_1 x_4}{x_2 x_3}} , \quad (78)$$

since  $K$  is defined as a positive constant.

Equations (56b) and (76b) become identical if

$$\begin{aligned} \frac{v_a}{K k_a} \left( (n_0 + 1)^2 + k_a (n_0 - 1)^2 \right) \left[ \phi(s) + \frac{1}{K n_0} \Phi(s) \right] \\ = (x_2 + x_3) \left[ \phi(s) + \frac{x_2 x_3}{x_1 x_4} \frac{x_1 + x_4}{x_2 + x_3} \Phi(s) \right] . \end{aligned} \quad (79)$$

By the expressions in the brackets in Eq. (79) we find immediately that by Eq. (78)

$$n_0 = \frac{x_2 + x_3}{x_1 + x_4} \sqrt{\frac{x_1 x_4}{x_2 x_3}} . \quad (80)$$

Equations (56c) and (76c) become identical if

$$\begin{aligned}
& \frac{1+k_a}{v_a} \left| \frac{1}{n_0} \phi(s) + \frac{1}{K} \Phi(s) \right| \\
&= \frac{x_1 + x_4}{x_1 x_4} \phi(s) + \frac{x_2 + x_3}{x_1 x_4} \Phi(s) \\
&= \frac{x_1 + x_4}{x_2 + x_3} \sqrt{\frac{x_2 x_3}{x_1 x_4}} \frac{x_2 + x_3}{\sqrt{x_1 x_2 x_3 x_4}} \phi(s) + \sqrt{\frac{x_2 x_3}{x_1 x_4}} \frac{x_2 + x_3}{\sqrt{x_1 x_2 x_3 x_4}} \Phi(s) \\
&= \frac{x_2 + x_3}{\sqrt{x_1 x_2 x_3 x_4}} \left[ \frac{x_1 + x_4}{x_2 + x_3} \sqrt{\frac{x_2 x_3}{x_1 x_4}} \phi(s) + \sqrt{\frac{x_2 x_3}{x_1 x_4}} \Phi(s) \right]. \quad (81)
\end{aligned}$$

Comparing the expressions in the brackets in Eq. (81) proves that Eqs. (78) and (80) are correct. Equation (81) also shows that

$$\frac{1+k_a}{v_a} = \frac{x_2 + x_3}{\sqrt{x_1 x_2 x_3 x_4}}. \quad (82)$$

By Eqs. (78) and (80)

$$K n_0 = \frac{x_2 + x_3}{x_1 + x_4} \frac{x_1 x_4}{x_2 x_3}. \quad (83)$$

Using the result in Eq. (83) we find that Eqs. (56a) and (76a) become identical if

$$n_0^2 + 1 + (n_0 - 1)^2 k_a = \frac{x_1 x_2 + x_3 x_4}{x_2 x_3} \frac{x_2 + x_3}{x_1 + x_4}, \quad (84)$$

and that Eqs. (56d) and (76d) become identical if

$$n_0^2 + 1 + \frac{(n_0 + 1)^2}{k_a} = \frac{x_1 x_3 + x_2 x_4}{x_2 x_3} \frac{x_2 + x_3}{x_1 + x_4}. \quad (85)$$

Note that  $(n_0 + 1)^2 = (n_0^2 + 1) + 2 n_0$ . Hence with Eq. (80), Eq. (84) can also be written as

$$(n_0 + 1)^2 + (n_0 - 1)^2 k_a = \frac{x_2 + x_3}{(x_1 + x_4)\sqrt{x_2 x_3}} \left[ \frac{x_1 x_2 + x_3 x_4}{\sqrt{x_2 x_3}} + 2\sqrt{x_1 x_4} \right] . \quad (86)$$

By Eq. (79)

$$\frac{v_a}{k_a} = \frac{x_2 + x_3}{(n_0 + 1)^2 + (n_0 - 1)^2 k_a} . \quad (87)$$

Therefore with Eq. (78)

$$\begin{aligned} \frac{k_a}{v_a} &= \frac{(n_0 + 1)^2 + (n_0 - 1)^2 k_a}{x_2 + x_3} \sqrt{\frac{x_2 x_3}{x_1 x_4}} = \frac{1}{x_1 + x_4} \left[ \frac{x_1 x_2 + x_3 x_4}{\sqrt{x_1 x_2 x_3 x_4}} + 2 \right] \\ &= \frac{(\sqrt{x_1 x_2} + \sqrt{x_3 x_4})^2}{(x_1 + x_4) \sqrt{x_1 x_2 x_3 x_4}} \end{aligned} . \quad (88)$$

By Eq. (82)

$$\frac{1}{v_a} + \frac{k_a}{v_a} = \frac{x_2 + x_3}{x_1 x_2 x_3 x_4} , \text{ so that with Eq. (88)}$$

$$\frac{1}{v_a} = \frac{(x_2 + x_3)(x_1 + x_4) - (x_1 x_2 + x_3 x_4) - 2\sqrt{x_1 x_2 x_3 x_4}}{(x_1 + x_4)\sqrt{x_1 x_2 x_3 x_4}} ,$$

which yields

$$v_a = \frac{(x_1 + x_4)\sqrt{x_1 x_2 x_3 x_4}}{(\sqrt{x_1 x_3} - \sqrt{x_2 x_4})^2} . \quad (89)$$

Then, Eq. (88) yields

$$k_a = \frac{(\sqrt{x_1 x_2} + \sqrt{x_3 x_4})^2}{(\sqrt{x_1 x_3} - \sqrt{x_2 x_4})^2} . \quad (90)$$

It can easily be shown that with the results obtained so far, all comparisons between Eqs. (56a, ..., e) and (76a, ..., e) are consistent. When the lattice two-port is known by the constants  $x_1, \dots, x_4$ , we are able to compute the constants  $K$ ,  $n_0$ ,  $v_a$ , and  $k_a$  which determine the equivalent tandem in the sequence where a pcra T of type (A) or type (B) is followed by a pcra T of type (C). Eq. (78) gives  $K$ , Eq. (80) gives  $n_0$ , Eq. (89) gives  $v_a$ , and Eq. (90) gives  $k_a$ . In the anticipated sequence of the sections in the tandem the ratio constants are  $n_a = 1/\bar{n}_b = n_0$ .

It is unnecessary to show the equivalence between a lattice two-port and a tandem in which a pcra T of type (C) is followed by a pcra T of type (B) or of type (A) by again equating the Eqs. (66a, ..., e) with the Eqs. (76a, ..., e). The constants  $K$  and  $n_0$  in this structure are given by the same Eqs. (78) and (80). By Eq. (67)

$$k_b = \frac{1}{k_a} \left[ \frac{n_0 + 1}{n_0 - 1} \right]^2 .$$

By Eq. (86)

$$1 + \frac{(n_0 - 1)^2}{(n_0 + 1)^2} k_a = 1 + \frac{1}{k_b} = \frac{1}{(n_0 + 1)^2} \frac{(x_2 + x_3)(\sqrt{x_1 x_2} + \sqrt{x_3 x_4})^2}{(x_1 + x_4)x_2 x_3} . \quad (91)$$

By Eq. (80) we find that

$$(n_0 + 1)^2 = \frac{|(x_2 + x_3)\sqrt{x_1 x_4} + (x_1 + x_4)\sqrt{x_2 x_3}|^2}{(x_1 + x_4)^2 x_2 x_3} .$$

Hence,

$$1 + \frac{1}{k_b} = (x_1 + x_4)(x_2 + x_3) \frac{(\sqrt{x_1 x_2} + \sqrt{x_3 x_4})^2}{|(x_2 + x_3)\sqrt{x_1 x_4} + (x_1 + x_4)\sqrt{x_2 x_3}|^2} ,$$

and

$$k_b = \frac{|(x_2 + x_3)\sqrt{x_1 x_4} + (x_1 + x_4)\sqrt{x_2 x_3}|^2}{(x_1 + x_4)(x_2 + x_3)(\sqrt{x_1 x_2} + \sqrt{x_3 x_4})^2} - \frac{|(x_2 + x_3)\sqrt{x_1 x_4} + (x_1 + x_4)\sqrt{x_2 x_3}|^2}{|(x_2 + x_3)\sqrt{x_1 x_4} + (x_1 + x_4)\sqrt{x_2 x_3}|^2} .$$

After some computation we obtain

$$\begin{aligned}
 k_b &= \frac{[(x_2 + x_3)\sqrt{x_1 x_4} + (x_1 + x_4)\sqrt{x_2 x_3}]^2}{(x_1 x_2 - x_3 x_4)^2} \\
 &= \frac{(\sqrt{x_1 x_3} + \sqrt{x_2 x_4})^2}{(\sqrt{x_1 x_2} - \sqrt{x_3 x_4})^2} .
 \end{aligned} \tag{92}$$

By Eq. (68)

$$v_d = v_a \frac{1 + k_b}{1 + k_a} .$$

By Eq. (92)

$$1 + k_b = \frac{(x_1 + x_4)(x_2 + x_3)}{(\sqrt{x_1 x_2} - \sqrt{x_3 x_4})^2} . \tag{93}$$

By Eq. (90)

$$1 + k_a = \frac{(x_1 + x_4)(x_2 + x_3)}{(\sqrt{x_1 x_3} - \sqrt{x_2 x_4})^2} . \tag{94}$$

By Eqs. (89) and (68)

$$v_d = \frac{(x_1 + x_4)\sqrt{x_1 x_2 x_3 x_4}}{(\sqrt{x_1 x_2} - \sqrt{x_3 x_4})^2} . \tag{95}$$

When the lattice two-port is known by its constants  $x_1, \dots, x_4$ , we are now able to compute directly the constants of the equivalent perfectly matched tandem in which a pcesa T of type (C) is followed in the forward direction by a pcesa T of type (B) or of type (A). The constants  $K$  and  $n_0$  are given by Eqs. (78) and (80). The ratio constants in the tandem are  $\bar{n}_c = 1/n_d = n_0$ . The constant  $k_b$  is given by Eq. (92) and the constant  $v_d$  by Eq. (95). Thus the tandem is known completely.

From a computational point of view the use of Eqs. (89), (90), (92), and (95) is not very practical since these equations necessitate the computation of some square

roots. Knowing  $K$  and  $n_0$  it is preferable to compute

$$k_a = \frac{1}{(n_0 - 1)^2} \left[ \frac{x_1 x_2 + x_3 x_4}{x_2 x_3} - \frac{n_0}{K} - (n_0^2 + 1) \right] , \quad (96)$$

and, respectively,

$$\frac{1}{k_b} = \frac{1}{(n_0 + 1)^2} \left[ \frac{x_1 x_2 + x_3 x_4}{x_2 x_3} - \frac{n_0}{K} - (n_0^2 + 1) \right] , \quad (97)$$

instead of using Eqs. (90) and (92).

Instead of using Eqs. (89) and (95) we compute by Eq. (87)

$$v_a = \frac{(x_2 + x_3) K k_a}{(n_0 + 1)^2 + (n_0 - 1)^2 k_a} , \quad (98)$$

and similarly we obtain

$$v_d = \frac{(x_2 + x_3) K k_b}{(n_0 + 1)^2 + (n_0 - 1)^2 k_b} . \quad (99)$$

Equations (98) and (99) can also be verified.

More important than the derivation of the equivalent perfectly matched tandems from the lattice two-port is the derivation of the lattice from the tandems. This reverse derivation can most elegantly be performed in the following way:

Consider Eq. (56b) and let us introduce

$$S = \frac{v_a}{k_a} \left[ (n_0 + 1)^2 + k_a (n_0 - 1)^2 \right] . \quad (100)$$

Since Eq. (56b) is identical with Eq. (76b),

$$S = (x_2 + x_3) K . \quad (101)$$

But, Eq. (56b) is also identical with Eq. (66b). Hence

$$S = \frac{v_d}{k_b} \left[ (n_0 + 1)^2 + k_b (n_0 - 1)^2 \right] . \quad (102)$$

By Eq. (83),

$$S = (x_1 + x_4)n_0 . \quad (103)$$

Consider now Eq. (56c) and let us introduce

$$P = \frac{1 + k_a}{v_a} K . \quad (104)$$

Since Eq. (56c) is identical with Eq. (76c), also

$$P = \frac{x_2 + x_3}{x_2 x_3} . \quad (105)$$

But Eq. (56c) is also identical with Eq. (66c), so that

$$P = \frac{1 + k_b}{v_d} K . \quad (106)$$

By Eq. (83)

$$P = \frac{x_1 + x_4}{x_1 x_4} K n_0 . \quad (107)$$

Let us now compute the product  $SP$ . By Eqs. (103) and (107) we obtain

$$SP = \frac{(x_1 + x_4)^2 K n_0^2}{x_1 x_4} . \quad (108)$$

By Eqs. (101) and (105) we obtain

$$SP = \frac{(x_2 + x_3)^2 K}{x_2 x_3} . \quad (109)$$

Let us further compute using the result obtained in Eq. (108):

$$+ \sqrt{1 - \frac{4Kn_0^2}{SP}} = \sqrt{1 - \frac{4x_1 x_4}{(x_1 + x_4)^2}} = \frac{x_1 - x_4}{x_1 + x_4} .$$

Hence

$$1 + \frac{x_1 - x_4}{x_1 + x_4} = \frac{2x_1}{x_1 + x_4} , \quad \text{and}$$

$$1 - \frac{x_1 - x_4}{x_1 + x_4} = \frac{2x_4}{x_1 + x_4} .$$

Note that by Eq. (103)

$$x_1 + x_4 = \frac{S}{n_0} ,$$

so that

$$x_1 = \frac{S}{2n_0} \left[ 1 + \sqrt{1 - \frac{4Kn_0^2}{SP}} \right] , \quad (110a)$$

and

$$x_4 = \frac{S}{2n_0} \left[ 1 - \sqrt{1 - \frac{4Kn_0^2}{SP}} \right] . \quad (110b)$$

Equations (110a, b) can briefly be written as

$$x_{1,4} = \frac{S}{2n_0} \left[ 1 \pm \sqrt{1 - \frac{4Kn_0^2}{SP}} \right] , \quad (111)$$

where the + sign refers to  $x_1$  and the - sign refers to  $x_4$ . Note that automatically by Eq. (110a)  $x_1 > x_4$ .

Using the result obtained in Eq. (109) and with so far undecided polarities let us compute:

$$\pm \sqrt{1 - \frac{4K}{SP}} = \pm \sqrt{1 - \frac{4x_2 x_3}{(x_2 + x_3)^2}} = \pm \frac{x_2 - x_3}{x_2 + x_3} .$$

Intermediately let the + sign be true, assuming that  $x_2 > x_3$ . Then

$$1 + \frac{x_2 - x_3}{x_2 + x_3} = \frac{2x_2}{x_2 + x_3} , \text{ and}$$

$$1 - \frac{x_2 - x_3}{x_2 + x_3} = \frac{2x_3}{x_2 + x_3} .$$

Note that by Eq. (101)

$$x_2 + x_3 = \frac{S}{K} ,$$

so that

$$x_2 = \frac{S}{2K} \left[ 1 + \sqrt{1 - \frac{4K}{SP}} \right] , \quad (112a)$$

and

$$x_3 = \frac{S}{2K} \left[ 1 - \sqrt{1 - \frac{4K}{SP}} \right] . \quad (112b)$$

If in our ambiguity the - sign holds, if  $x_3 > x_2$ , then

$$x_2 = \frac{S}{2K} \left[ 1 - \sqrt{1 - \frac{4K}{SP}} \right] , \quad (113a)$$

$$x_3 = \frac{S}{2K} \left[ 1 + \sqrt{1 - \frac{4K}{SP}} \right] . \quad (113b)$$

We now have to find a condition that decides whether for a given tandem the pair of Eqs. (112a,b) or of Eqs. (113a,b) is the correct one in order to obtain the impedance branches  $X_2$  and  $X_3$  of the equivalent lattice.

Note again that Eqs. (76a) and (56a) are identical if

$$\frac{x_1 x_2 + x_3 x_4}{x_1 x_4} = \frac{1}{K n_0} \left[ (n_0^2 + 1) + k_a (n_0 - 1)^2 \right] , \quad (114)$$

and Eqs. (76d) and (56d) are identical if

$$\frac{x_1 x_3 + x_2 x_4}{x_1 x_4} = \frac{1}{K n_0} \left[ (n_0^2 + 1) + \frac{(n_0 + 1)^2}{k_a} \right] . \quad (115)$$

Then

$$\frac{x_1 x_2 + x_3 x_4}{x_1 x_4} - \frac{x_1 x_3 + x_2 x_4}{x_1 x_4} = \frac{(x_1 - x_4)(x_2 - x_3)}{x_1 x_4} . \quad (116)$$

Since we assumed that  $x_1 > x_4$ , the result in Eq. (116) is positive if  $x_2 > x_3$ .

On the other hand

$$\begin{aligned} \frac{(x_1 - x_4)(x_2 - x_3)}{x_1 x_4} &= \frac{1}{K n_0} \left[ k_a (n_0 - 1)^2 - \frac{(n_0 + 1)^2}{k_a} \right] \\ &= \frac{(n_0 + 1)^2}{K k_a n_0} \left[ k_a^2 \frac{(n_0 - 1)^2}{(n_0 + 1)^2} - 1 \right] . \end{aligned} \quad (117)$$

The final result in Eq. (117) can only be positive if

$$k_a > \left| \frac{n_0 + 1}{n_0 - 1} \right| . \quad (118)$$

But then by Eq. (67) simultaneously

$$k_b < \left| \frac{n_0 + 1}{n_0 - 1} \right| . \quad (119)$$

If  $x_3 > x_2$ , then the result in Eq. (116) is negative and consequently the opposite signs of inequality are true in Eqs. (118) and (119). Hence, we can combine Eqs. (112a, b) and/or Eqs. (113a, b) in

$$x_{2,3} = \frac{S}{2K} \left[ 1 \pm \sqrt{1 - \frac{4K}{SP}} \right], \quad (120)$$

where the + sign refers to  $x_2$  and the - sign refers to  $x_3$  when

$$k_a > \left| \frac{n_0 + 1}{n_0 - 1} \right| > k_b \quad (120a)$$

and where the + sign refers to  $x_3$  and the - sign refers to  $x_2$  when

$$k_a < \left| \frac{n_0 + 1}{n_0 - 1} \right| < k_b \quad (120b)$$

We are now able to state the following theorem:

**THEOREM 4** (concerning the equivalence between a perfectly matched tandem and a lattice two-port)

A perfectly matched tandem of pcsa T's where a section of either type (A) or type (B) is followed by a section of type (C), or a tandem where a section of type (C) is followed by a section of either type (B) or type (A) is equivalent with a lattice two-port. The lattice implies in one pair of its branches the normalized positive real frequency function  $\phi(s)$  and in the other pair the function  $\Phi(s)$  as indicated in Figure 9.

Assume now that one of the tandems is known and we want to find the constants which determine the lattice. We compute as follows:

Depending which section sequence (either (AB)C or C(BA)) is given, we compute S and P by Eqs. (100) and (104) or by Eqs. (102) and (106), respectively. Then we compute  $x_1$  and  $x_4$  by Eq. (111) in which the indices are clearly identified. We know  $n_0$  by Eqs. (55) or (65), respectively; hence, we can compute  $\frac{n_0 + 1}{n_0 - 1}$  and we can compare this quotient with the known  $k_a$  or  $k_b$ . This enables us to decide about the relation between the signs of polarity and the components  $x_2$  and  $x_3$  in Eq. (120) according to the relations in Eq. (120a, b). Eq. (120) can now be evaluated.

In general, the lattice two-port is not a symmetrical one. It becomes symmetrical if  $x_1 = x_4$  and  $x_2 = x_3$ . This is the case when Eqs. (71) and (72) hold.

The stated equivalence is of extreme importance. It allows us to find a very simple and transformerless realization for the otherwise highly complicated perfectly matched tandem. One will agree, however, that a tandem in which the sections are perfectly matched is a rare situation. More often, supposedly, we meet a tandem in which one section is of type (C) and the other of type (A) or of type (B), where the sections are matched ( $v/x = \bar{v}/\bar{x}$ ), but not perfectly matched. It has been shown in Section 2 that if port impedance implying the frequency function  $\Phi(s)$  is available, one can change the ratio constant by transposing this impedance to the other port. In this situation, there is hope that only a matched tandem can be changed to a perfectly matched one. To investigate this is the aim of the next section.

#### 6. PERFECT MATCHING IN A TANDEM OBTAINED BY THE TRANSPOSITION OF PORT IMPEDANCE

Consider Figure 10 which shows two matched tandems terminated by port impedances. The T sections are matched ( $v_a/x_a = \bar{v}_b/\bar{x}_b$  and  $v'_a/x'_a = \bar{v}'_b/\bar{x}'_b$ ) in both tandems. The frequency functions  $\phi(s)$  and  $\Phi(s)$  are expressed in the familiar way of shading. The figure contains only the necessary notations of the impedance constants and of the ratio constants.

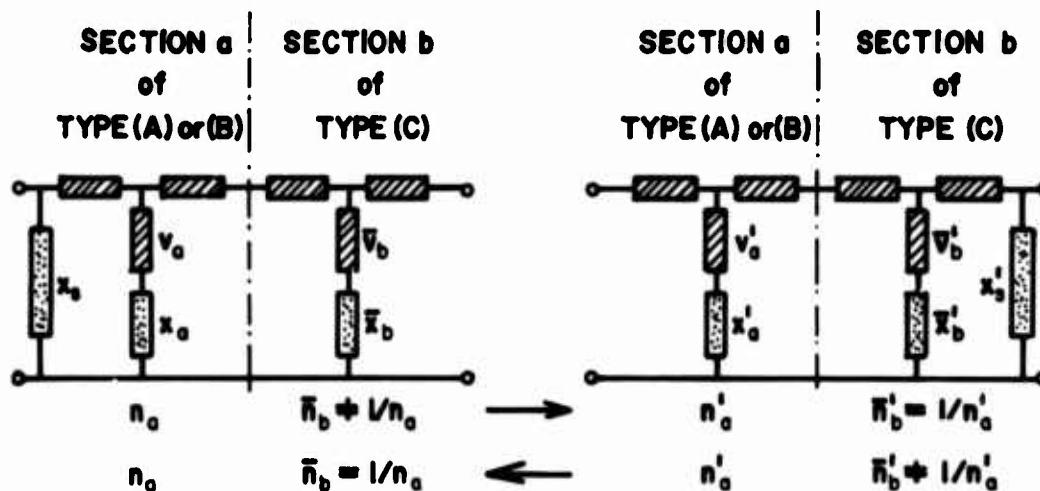


Figure 10. Impedance Transposition Between the Ports of a Tandem

Assume now that the left side two-port has an in-port impedance  $X_s = x_s \cdot \Phi(s)$  and assume further that its pccsa T sections are not perfectly matched ( $\bar{n}_b \neq 1/n_a$ ). We suppose from our discussions in Section 2 that we are able to transpose the in-port impedance to the out-port, where it appears as  $X'_s = x'_s \cdot \Phi(s)$

and we hope that we can find a proper magnitude  $x_s$  such that by the transposition the two-port becomes perfectly matched ( $\bar{n}_b' = 1/n_a'$ ). The constant  $x_s$  of course has to be a positive one. If we succeed in this attempt, then the perfectly matched two-port at the right in Figure 10 becomes equivalent with a lattice two-port; then

the two-port shown at the right in Figure 10 has a realization that is shown in Figure 11.

We transpose the in-port impedance (see the two-port at the left in Figure 10) in two steps. First we transpose it to the out-port of Section 2 where it appears as  $X_s''$ . By Eq. (32) we obtain

$$n_a' = n_a \frac{x_s}{x_s + x_a(n_a - 1)^2} . \quad (121)$$

Figure 11. Lattice Terminated at the Out-Port

By Eq. (33)

$$x_s'' = \frac{x_s}{n_a n_a'} . \quad (122)$$

By Eq. (34)

$$\frac{1}{x_a'} = \frac{1}{x_a} + \frac{1}{x_s} - \frac{1}{x_s''} . \quad (123)$$

By Eq. (35)

$$v_a' = v_a \frac{x_a'}{x_a} . \quad (124)$$

We now perform the second step of the transposition. We transpose  $X_s''$  to the out-port where it appears as  $X_s'$ . By Eq. (43)

$$\bar{n}_b' = \bar{n}_b \frac{x_s''}{x_s'' + \bar{x}_b(\bar{n}_b + 1)^2} . \quad (125)$$

By Eq. (44)

$$x_s' = \frac{x_s''}{\bar{n}_b \bar{n}_b'} . \quad (126)$$

By Eq. (45)

$$\frac{1}{\bar{x}'_b} = \frac{1}{\bar{x}_b} + \frac{1}{x''_s} - \frac{1}{x'_s} . \quad (127)$$

By Eq. (46)

$$\bar{v}'_b = \bar{v}_b \frac{\bar{x}'_b}{\bar{x}_b} . \quad (128)$$

We postulate that

$$n'_a = \frac{1}{\bar{n}'_b} = n_a \frac{x_s}{x_s + x_a(n_a - 1)^2} . \quad (129)$$

By substituting  $x''_s$  from Eq. (122) into Eq. (125) we obtain

$$\bar{n}'_b = \bar{n}_b \frac{x_s}{x_s + \bar{x}_b n_a n'_a (\bar{n}_b + 1)^2} . \quad (130)$$

By substituting  $n'_a$  from Eq. (121) in Eq. (130) we obtain

$$\bar{n}'_b = \bar{n}_b \frac{x_s + x_a(n_a - 1)^2}{x_s + x_a(n_a - 1)^2 + \bar{x}_b n_a (\bar{n}_b + 1)^2} . \quad (131)$$

By Eq. (129) we obtain the following equation in which  $x_s$  is the unknown:

$$x_s n_a \bar{n}_b = x_s + x_s(n_a - 1)^2 + \bar{x}_b n_a^2 (\bar{n}_b + 1)^2 .$$

Solved for  $x_s$

$$x_s = \frac{1}{n_a \bar{n}_b - 1} \left[ x_a(n_a - 1)^2 + \bar{x}_b n_a^2 (\bar{n}_b + 1)^2 \right] . \quad (132)$$

Recall that  $x_s$  has to be positive. The expression in the brackets in Eq. (132) is definitely positive, since  $x_a$  and  $\bar{x}_b$  are defined as positive constants of impedance

character. The denominator  $n_a \bar{n}_b$ , however, is only positive if

$$n_a \bar{n}_b > 1. \quad (133)$$

Therefore, the success of obtaining a perfectly matched tandem depends not only on the magnitude of  $x_s$  but also on the ratio constants  $n_a$  and  $\bar{n}_b$  before the transposition. Only if the inequality in Eq. (133) holds, the aim of the transposition can be obtained. Evidently, the limit  $n_a \bar{n}_b = 1$  postulates  $x_s \rightarrow \infty$ . In this event the sections are already perfectly matched and no in-port impedance has to be transposed.

Assume now that we have a tandem of matched (but not perfectly matched) sections in the type sequence (AB)C and we ask: what in-port impedance  $X_s$  is necessary to obtain perfect matching by the impedance transposition? To answer the question we first have to check whether or not the inequality (133) holds. If it does not hold, we know that perfect matching cannot be obtained. If it holds, we compute  $x_s$  by Eq. (132). The constants after the transposition are known by the respective equations in the series of Eqs. (121) through (128). The constants  $x'_1, \dots, x'_4$  determining the lattice in Figure 11 are obtained by Eqs. (111) and (120) in the preceding section. The lattice two-port, of course, has a termination impedance  $X'_s = x'_s \cdot \Phi(s)$ .

Next assume that (with other values of the constants than before) a tandem such as shown at the right in Figure 10 is matched, but not perfectly matched. Let the tandem be terminated by the out-port impedance  $X'_s$  and let this impedance be such that by transposing it to the in-port the tandem becomes perfectly matched. Let us now answer the question: what constant  $x'_s$  is necessary that perfect matching can be achieved.

In the backward direction we transpose  $X'_s$  in two steps. First we transpose it from the out-port to the in-port of section b in the right part of Figure 10. It appears there as  $X''_s$ . We apply in sequence Eqs. (39) through (42) and obtain:

$$\bar{n}_b = \bar{n}'_b + \frac{(\bar{n}'_b + 1)^2}{\bar{n}'_b} \frac{\bar{x}'_b}{x'_s}, \quad (134)$$

$$x''_s = x'_s \bar{n}_b \bar{n}'_b, \quad (135)$$

$$\frac{1}{\bar{x}_b} = \frac{1}{\bar{x}'_b} + \frac{1}{x'_s} - \frac{1}{x''_s}, \quad (136)$$

$$\bar{v}_b = \bar{v}'_b \frac{\bar{x}_b}{\bar{x}'_b} . \quad (137)$$

Next we transpose  $x''_s$  to the in-port of the tandem where it appears as  $x_s'$  in the left part of Figure 10. We apply Eqs. (25), (28), (29), and (30) in sequence and we obtain:

$$n_a = n'_a + \frac{(n'_a - 1)^2}{n'_a} \frac{x'_a}{x''_s} , \quad (138)$$

$$x_s' = x''_s n_a n'_a , \quad (139)$$

$$\frac{1}{x_a} = \frac{1}{x'_a} + \frac{1}{x''_s} - \frac{1}{x_s'} , \quad (140)$$

$$v_a = v'_a \frac{x_a}{x'_a} . \quad (141)$$

We postulate that

$$n_a = 1 / \bar{n}_b \quad (142)$$

and we obtain that the necessary  $x'_s$  is given by

$$x'_s = \frac{1}{n'_a \bar{n}'_b (1 - n'_a \bar{n}'_b)} \left[ x'_a (n'_a - 1)^2 + \bar{x}'_b n'_a^2 (\bar{n}'_b + 1)^2 \right] . \quad (143)$$

The constant  $x'_s$  has to be positive. In Eq. (143) the expression within the brackets is definitely positive since  $x'_a$  and  $\bar{x}'_b$  are defined as positive. Hence  $x'_s$  is positive when the denominator in Eq. (143) is positive. Therefore, in order to make  $x'_s$  positive it is necessary that

$$n'_a \bar{n}'_b < 1 \quad (144)$$

before the transposition.

When we intend to match perfectly a tandem that is shown at the right in Figure 10, we check first whether or not the inequality (144) holds. If it does not hold, we know that perfect matching cannot be achieved. If it holds, we compute

the necessary  $x'_s$  by Eq. (143) and the constants after the transposition by Eqs. (134) through (141). The perfectly matched tandem that has the termination  $X_s$  at its in-port as shown at the left in Figure 10 has an equivalent lattice structure as shown in Figure 12. The lattice of course has the impedance  $X_s$  at its in-port. Its constants  $x_1, \dots, x_4$  are obtained by Eqs. (111) and (120).

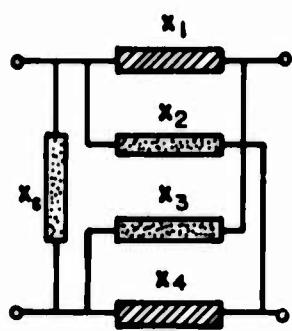


Figure 12. Lattice Terminated at the In-Port

For completeness we will now assume that the sections appear in the opposite sequence. First assume that, as shown in the left part of Figure 13 a pcsa T section of type (C) is followed by a Section d that is of type (A) or of type (B). Let the sections be matched (but not perfectly matched). Assume that  $n_d \neq 1/\bar{n}_c$ . What is the in-port impedance  $X_s$  that has to be transposed to the out-port where it appears as  $X'_s$  so that perfect matching ( $n'_d = 1/\bar{n}'_c$ ) is obtained?

We first transpose  $X_s$  over Section c. By Eqs. (43), ..., (46) we obtain:

$$\bar{n}'_c = \bar{n}_c \frac{x_s}{x_s + \bar{x}_c (\bar{n}_c + 1)^2}, \quad (145)$$

$$x''_s = \frac{x_s}{\bar{n}_c \bar{n}'_c}, \quad (146)$$

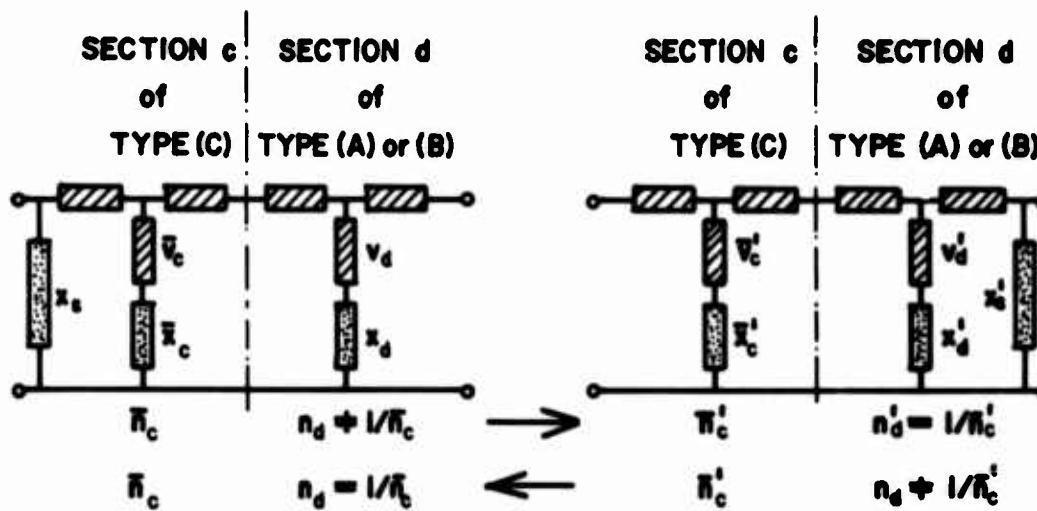


Figure 13. Impedance Transposition Between the Ports of a Tandem of the Reverse Sequence Compared With Figure 10

$$\frac{1}{\bar{x}'_c} = \frac{1}{\bar{x}_c} + \frac{1}{x_s} - \frac{1}{x''_s} \quad , \quad (147)$$

$$\bar{v}'_c = \bar{v}_c \frac{\bar{x}'_c}{\bar{x}_c} \quad . \quad (148)$$

Next we transpose  $x''_s$  to the out-port where it appears as  $x'_s$ . By Eqs. (25), (28), (29), and (30) we obtain:

$$n'_d = n_d \frac{x''_s}{x''_s + x_d(n_d - 1)^2} \quad , \quad (149)$$

$$x'_s = \frac{x''_s}{n_d n'_d} \quad , \quad (150)$$

$$\frac{1}{x'_d} = \frac{1}{x_d} + \frac{1}{x''_s} - \frac{1}{x'_s} \quad , \quad (151)$$

$$v'_d = v_d \frac{x'_d}{x_d} \quad . \quad (152)$$

In order to make

$$n'_d = 1/\bar{n}_c \quad (153)$$

it is necessary that

$$x_s = \frac{1}{\bar{n}_c n_d - 1} \left[ \bar{x}_c (\bar{n}_c + 1)^2 + x_d \bar{n}_c^2 (n_d - 1)^2 \right] \quad . \quad (154)$$

The constant  $x_s$  is positive only when

$$\bar{n}_c n_d > 1 \quad . \quad (155)$$

The inequality (155) shows that perfect matching can only be obtained when, before the transposition in the forward direction, the product of the ratio constants is

greater than 1. This is qualitatively the same postulation as we met in (133) in performing the first forward transposition.

If in a tandem which is not perfectly matched and where a type (C) section is followed by a type (A) or (B) section the inequality (155) holds, an in-port impedance  $X_s$ , given by the constant  $x_s$  in Eq. (154), can be transposed to the out-port. The constants after the transposition can be computed by Eqs. (145) through (148) and Eqs. (149) through (152). The perfectly matched tandem obtained after the transposition is equivalent to a lattice such as shown in Figure 11. The lattice is terminated by the transposed  $X'_s$ ; its constants can be computed by Eqs. (111) and (120).

Finally, consider the tandem that is shown at the right in Figure 13. Assume that (with values other than those used previously) the Sections c and d in this tandem are not perfectly matched ( $n_d \neq 1/\bar{n}_c'$ ). What is the out-port impedance  $X'_s$  which has to be transposed to the in-port so that perfect matching ( $n_d = 1/\bar{n}_c'$ ) is achieved?

We transpose  $X'_s$  from the out-port over Section d and we obtain by Eqs. (24), (28), (29), and (30)

$$n_d = n'_d + \frac{(n'_d - 1)^2}{n'_d} \frac{x'_d}{x'_s} , \quad (156)$$

$$x''_s = x'_s n_d n'_d , \quad (157)$$

$$\frac{1}{x_d} = \frac{1}{x'_d} + \frac{1}{x'_s} - \frac{1}{x''_s} , \quad (158)$$

$$v_d = v'_d \frac{x_d}{x'_d} . \quad (159)$$

Next we transpose  $X''_s$  over Section c to the in-port. We apply Eqs. (39) through (42) to obtain:

$$\bar{n}_c = \bar{n}'_c + \frac{(\bar{n}'_c + 1)^2}{\bar{n}'_c} \frac{\bar{x}'_c}{x''_s} , \quad (160)$$

$$x_s = x''_s \bar{n}_c \bar{n}'_c , \quad (161)$$

$$\frac{1}{\bar{x}_c} = \frac{1}{\bar{x}'_c} + \frac{1}{\bar{x}''_s} - \frac{1}{x_s} , \quad (162)$$

$$\bar{v}_c = \bar{v}'_c \frac{\bar{x}_c}{\bar{x}'_c} . \quad (163)$$

Postulating that

$$n_d = 1/\bar{n}_c \quad (164)$$

yields

$$x'_s = \frac{1}{\bar{n}'_c \bar{n}'_d (1 - \bar{n}'_c \bar{n}'_d)} \left[ \bar{x}'_c (\bar{n}'_c + 1)^2 + x'_d \bar{n}'_c^2 (n'_d - 1)^2 \right] . \quad (165)$$

The constant  $x'_s$  is only positive when

$$\bar{n}'_c \bar{n}'_d < 1 . \quad (166)$$

When the inequality (166) holds, the necessary constant  $x'_s$  of the impedance to be transposed from the out-port to the in-port can be computed by Eq. (165). The constants of the tandem after the transposition are obtained by Eqs. (156) through (163). After the transposition the perfectly matched tandem has an equivalent lattice structure as shown in Figure 12. The constants of the lattice are obtained by Eqs. (111) and (120). The lattice is terminated at its in-port by the impedance  $X_s$ .

Note that the inequality postulation (166) demands that the product of the ratio constants before the transposition be smaller than 1 in the second backward transposition, also. Hence, it is common to the forward transposition that this product is greater than 1 and to the backward transposition that this product is smaller than 1.

We state the following theorem:

**THEOREM 5** (concerning the transformation of a matched tandem into a perfectly matched one)

A simply (not perfectly) matched tandem can be transformed into a perfectly matched tandem by a forward or a backward transposition of port impedance if before the transposition the magnitude of the product of the ratio constants is greater than 1

for an anticipated forward transposition, and if the magnitude of this product is smaller than 1 for an anticipated backward transposition.

For practical purposes it is not sufficient to answer only the question about the necessary magnitude  $x_s$  and  $x'_s$  of the impedances to be transposed. For instance, if we intend to transpose the impedance  $X_s$  in the forward direction, a sufficient amount of in-port impedance  $X_S$  must be available. If  $X_S = X_s$ , then the impedance is transposed totally. Assume now that

$$\frac{1}{X_S} = \frac{1}{X_s} + \frac{1}{X_r} . \quad (167)$$

In Eq. (167) the more general case is shown where the available port impedance  $X_S$  is split into two components. The component  $X_s$  is transposed and acts in performing perfect matching. The other component  $X_r$  remains at the in-port of the now perfectly matched tandem as well as at the in-port of the equivalent lattice. In order to achieve perfect matching, therefore, not only must the inequalities (133) and (155) hold before the transposition, but it is also necessary that

$$x_S < x_s \quad (168)$$

in order to make  $X_r$  in Eq. (167) positive. Likewise in the event of a backward transposition where the available out-port impedance is split according to

$$\frac{1}{X'_S} = \frac{1}{X'_s} + \frac{1}{X'_r} . \quad (169)$$

besides holding the inequalities (144) and (166), it is necessary that

$$x'_S < x'_s \quad (170)$$

in order to make  $X'_r$  positive in Eq. (169).

## 7. SOME PARTICULAR STRUCTURES OF A PERFECTLY COUPLED AND SHUNT-AUGMENTED T

In Section 1 where we defined the pcsa T, we assumed that the normalized frequency functions  $\phi(s)$  and  $\Phi(s)$  are unrelated and we only postulated that they be positive real functions. Let us retain this postulation for  $\Phi(s)$ , but let us assume that

$$\phi(s) = s . \quad (171)$$

Equation (171) presents the particular frequency function that causes the perfectly coupled branches of the T to become perfectly coupled inductive impedances. But we know by Section 1 that an inductive star, for which the sum of the inverse inductances disappears, has a perfectly coupled transformer as an equivalent circuit. There-

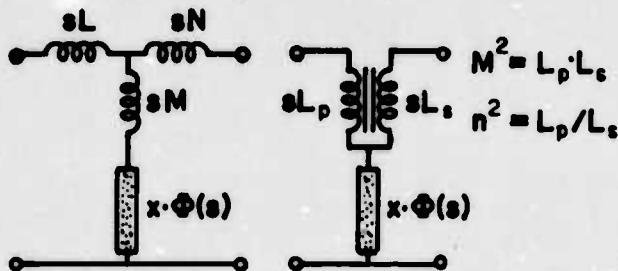


Figure 14. pcsa T With the Frequency Function  $\phi(s) = s$

can be transposed either totally or partially. Therefore, we have a certain freedom in this event in regard to the design of the transformer.

Assume that in addition to Eq. (171) the other frequency function

$$\Phi(s) = 1/s . \quad (172)$$

By Eq. (172) the shunt-augmentation becomes a capacitive impedance. Then, if Eqs. (171) and (172) hold, the pcsa T becomes the well known and classical Brune section which for a positive transformer ratio is shown in Figure 15. Since the publication of Otto Brune's famous paper (1931), the Brune section shown in Figure 15 has always been considered as a general configuration of a two-port section in network theory. By Eqs. (171) and (172), however, this section appears as a particularity of a much more general section, the pcsa T. A first step towards the generalization has stimulated the author to write a Letter-to-the-Editor (Haase, 1964) after having read Miyata's paper (1963). At that time the author thought it

therefore, a pcsa T in which  $\phi(s) = s$  can be realized by the circuits shown in Figure 14. In part (a) of Figure 14 we assume that the transformer ratio is positive, in part (b) we assume it as negative. The shunt-augmentation can be any positive real impedance function.

The case where  $\phi(s) = s$  is the only one where a pcsa T can be realized as such. If port impedance is available this impedance

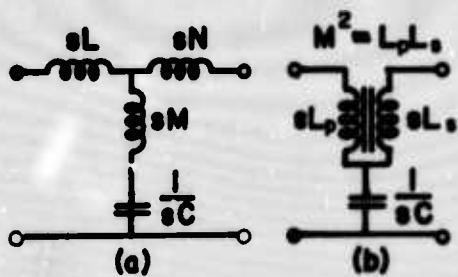


Figure 15. pcsa T of the Brune Type

A more or less trivial pcsa T of type (A) or type (B) is the one that degenerates to a shunt two-port by the particular ratio constant  $n = 1$ .

Since we know by Section 3 that by a forward transposition of a port impedance the ratio constant decreases, and by a backward transposition it increases, one may ask: is it possible to find a port impedance such that by a respective transposition a ratio coefficient  $n = 1$  can be achieved. This would be something similar with the achievement of perfect matching as discussed in the previous section. In the present question it would mean that a degeneration of a pcsa T of types (A) or (B) can be obtained. The answer to this question however is NO. Equations (25) and (32) show immediately that  $n = 1$  also postulates  $n' = 1$ ; hence, the ratios  $n$  and  $n'$  are as singular events always simultaneously equal to 1.

### 8. THE pcsa T WITH DUAL SHUNT COMPONENTS AND ITS REALIZATION

Assume now a pcsa T with the normalized frequency function  $\phi(s)$  in the perfectly coupled branches and let the other normalized frequency function be

$$\Phi(s) = 1/\phi(s) . \quad (173)$$

Of such a T we like to say that its shunt components are dual. When the one shunt component is of the impedance  $v \cdot \phi(s)$  and the other of the impedance  $x/\phi(s)$ , their product is  $v \cdot x$  that is the square of the duality constant. Figure 16 shows such a T in part (a) when it is of type (A) or (B) and in part (b) when it is of type (C).

Let the T be of type (B) with  $n > 1$  so that  $v(\frac{1}{n} - 1)\phi(s)$  is negative and  $v(n-1)\phi(s)$  is positive, and let the two-port be terminated by an impedance function  $Z_t(s)$ . This two-port is shown in Figure 17.

necessary to stick to Eq. (171) and that Eq. (172) seemed to be an unnecessary restriction. He later found that Eq. (171) can also be dropped if one disregards that if  $\phi(s) \neq s$ , but if  $\phi(s)$  and  $\Phi(s)$  are normalized positive real frequency functions, the general pcsa T cannot be realized as such; nevertheless it is a most valuable model of a two-port section.

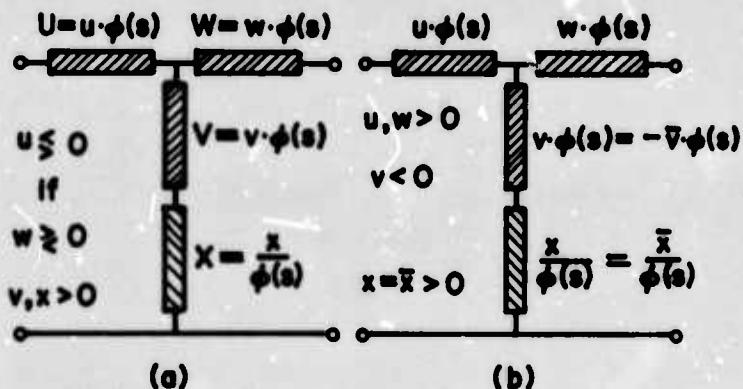


Figure 16. pcesa T of Type (A) or (B) in (a) and of Type (C) in (b) with Dual Shunt Components

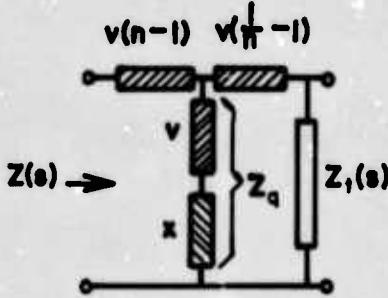


Figure 17. pcesa T With Dual Shunt Components Terminated by  $Z_t(s)$

Call the impedances

$$v(n-1) \phi(s) = Z_a(s) , \quad (174a)$$

$$v\left(\frac{1}{n}-1\right) \phi(s) = Z_b(s) , \quad (174b)$$

$$v \cdot \phi(s) + x/\phi(s) = Z_q(s) . \quad (174c)$$

Then the driving-point impedance measured at the in-port of the circuit in Figure 17 is

$$Z(s) = Z_a(s) + \frac{1}{\frac{1}{Z_q(s)} + \frac{1}{Z_b(s) + Z_t(s)}} . \quad (175)$$

Evidently, when at a frequency  $s = s_0$  the sum of the impedances

$$Z_b(s_0) + Z_t(s_0) = 0 . \quad (176)$$

Then

$$Z(s_0) = Z_a(s_0) . \quad (177)$$

Such a frequency  $s_0$  certainly exists; the sum in Eq. (176) implies the branch element  $Z_b(s)$  that is negative as we know. Hence, even  $s_0$  will be real and positive.

There is also a complex frequency  $s_q$  for which

$$Z_q(s_q) = 0, \quad (178)$$

so that

$$Z(s_q) = Z_a(s_q). \quad (179)$$

Let us summarize: whenever either  $Z_q(s)$  or the sum  $Z_b(s) + Z_t(s)$  in which  $Z_b(s)$  is a negative impedance, becomes zero, we measure  $Z_a(s)$  as the driving-point impedance of the two-port in Figure 17.

Consider now the lattice that is shown in the left part of Figure 18. It has the pair of branches  $T \cdot \phi(s)$  and  $T/\phi(s)$  and the other pair of branches  $T/F(s)$  and  $T \cdot F(s)$ .

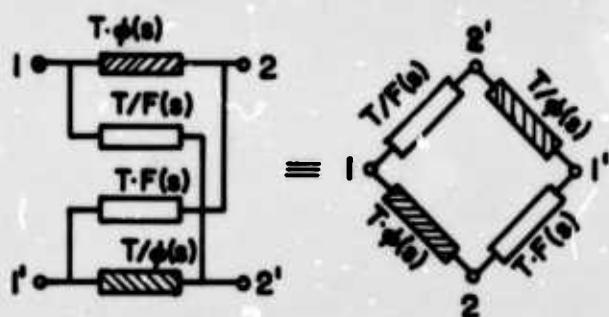


Figure 18. Balanced Bridge Structure

In the right part of Figure 18 the circuit is redrawn and shows a bridge representation that may be a more familiar picture of the circuit. One will recognize immediately that this bridge is balanced by the proper choice of the branch impedances. Hence, it does not matter whether or not this balanced bridge is terminated at the terminal pair 2 and 2'; we may even short circuit these terminals and thus obtain the circuit shown

in Figure 19. This circuit is equivalent in regard to its in-port impedance to the circuit in Figure 18. The correspondent pairs of branches in Figures 18 and 19 have the same constant of duality that is  $T$  since

$$|T \cdot \phi(s)| T/\phi(s) = T \cdot F(s) \cdot T/F(s) = T^2. \quad (180)$$

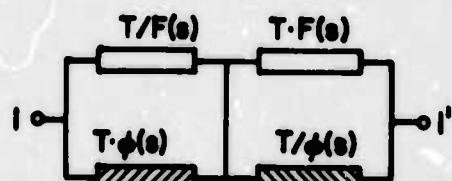


Figure 19. Bridge Equivalence With That of Figure 18

In order to relate the shortened lattice circuit with the pcsa  $T$  with dual shunt components, we will introduce somewhat different notations and, furthermore, refer to Figure 20. In Figure 20 the branches in the left part of the circuit have the impedances  $Z_a(s)$  and  $Z_0(s)$ . The corresponding

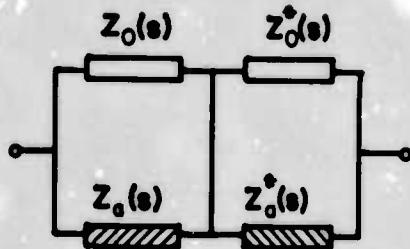


Figure 20. Realization of the Driving-Point Impedance of a pcsa T of Type (B)

branches in the right part of the circuit have the impedances

$$Z_a^*(s) = T^2/Z_a(s) , \quad (181a)$$

and

$$Z_0^*(s) = T^2/Z_0(s) . \quad (181b)$$

Thus, when for example  $Z_0^*(s) = 0$ , then

$Z_0(s) = \infty$ . Assume now that  $Z_0^*(s)$  is

simultaneously zero when  $Z_q(s)$  is zero and vice versa. This means that the zeros of  $Z_0^*(s)$  and those of  $Z_q(s)$  are the same.

The driving-point in-port impedance of the circuit in Figure 20 is

$$Z(s) = Z_1(s) + Z_2(s) , \quad (182)$$

where

$$\frac{1}{Z_1(s)} = \frac{1}{Z_a(s)} + \frac{1}{Z_0(s)} = \frac{Z_a(s) + Z_0(s)}{Z_a(s)Z_0(s)} , \quad (183a)$$

and

$$\frac{1}{Z_2(s)} = \frac{1}{Z_a^*(s)} + \frac{1}{Z_0^*(s)} = T^2 |Z_a(s) + Z_0(s)| . \quad (183b)$$

Hence

$$Z(s) = \frac{Z_a(s)Z_0(s) + T^2}{Z_a(s) + Z_0(s)} . \quad (184)$$

We measure  $Z(s) = Z_a(s)$  when either in Eq. (184)  $Z_0(s) = \infty$ , or for  $s_0$  as shown previously. Hence in Eq. (184)

$$T = Z_a(s_0) \quad (185)$$

where  $s_0$  is a real solution of the equation

$$Z_b(s) + Z_t(s) = 0 . \quad (186)$$

We now know  $Z(s)$ ,  $Z_a(s)$  and  $T$  in Eq. (184) and we are thus able to determine

$$Z_0(s) = \frac{T^2 - Z_a(s)Z(s)}{Z(s) - Z_a(s)} . \quad (187)$$

Equation (184) can also be written as

$$Z(s) = \frac{Z_a(s)Z_0(s)}{Z_a(s) + Z_0(s)} + \frac{T^2}{Z_a(s) + Z_0(s)} . \quad (184a)$$

Since  $Z(s)$  and  $Z_a(s)$  are positive real impedances and  $T$  is a positive constant,  $Z_0(s)$  must also be a positive real function. Thus,  $Z_0(s)$  and its dual  $Z_0^*(s)$  are realizable and with them the complete circuit in Figure 20. This circuit is similar with a Bott-Duffin (1949) circuit but is more general. It has the same driving-point impedance as the circuit shown in Figure 17, but otherwise it is not equivalent to this circuit.

We have shown in this section that a positive real impedance function can be realized as the driving-point impedance function of a transformerless and Bott-Duffin - like circuit. We assumed that the implied pcsa  $T$  was of type (B) and had dual shunt impedance circuits. It can be shown that a similar realization is also possible when the implied pcsa  $T$  is of type (A) or of type (C). It is not the purpose of this report to discuss all these realization problems; we intend to devote other reports to these problems in the near future. But, it has been felt worthwhile to mention here one of the realizations in which a pcsa  $T$  is implied.

## 9. THE DUAL TWO-PORT OF THE PERFECTLY COUPLED AND SHUNT-AUGMENTED T

We call two two-ports dual when the one is described by the Kirchhoff voltage equations in the same way as the other is described by the Kirchhoff current equations. As an example consider the circuits (a) and (b) in Figure 21. The circuit (a) in this figure is a T with the branch impedances  $X_1$ ,  $X_2$ , and  $X_3$ . It has the following elements of the chain matrix:

$$\hat{A}_T = X_1 + X_2 , \quad (188a)$$

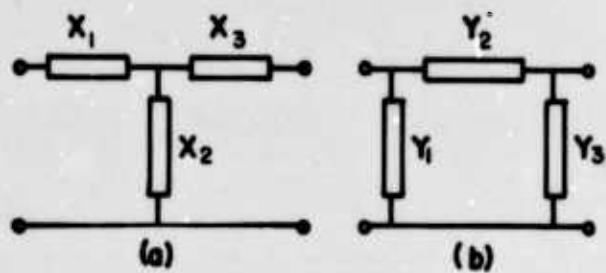


Figure 21. A T Two-Port (a) and Its Dual Pi Two-Port (b)

$$\hat{B}_T = X_1 X_2 + X_1 X_3 + X_2 X_3 , \quad (188b)$$

$$\hat{C}_T = 1 , \quad (188c)$$

$$\hat{D}_T = X_2 + X_3 , \quad (188d)$$

$$\hat{E}_T = X_2 . \quad (188e)$$

The circuit (b) in Figure 21 is a Pi with branch admittances  $Y_1$ ,  $Y_2$ , and  $Y_3$ . It has the following elements of the chain matrix:

$$\hat{A}_{Pi} = Y_2 + Y_3 , \quad (189a)$$

$$\hat{B}_{Pi} = 1 , \quad (189b)$$

$$\hat{C}_{Pi} = Y_1 Y_2 + Y_1 Y_3 + Y_2 Y_3 , \quad (189c)$$

$$\hat{D}_{Pi} = Y_1 + Y_2 , \quad (189d)$$

$$\hat{E}_{Pi} = Y_2 . \quad (189e)$$

Assume now that we make

$$X_1 = Y_1 , \quad (190a)$$

$$X_2 = Y_2 \quad (190b)$$

$$X_3 = Y_3 \quad (190c)$$

This is possible when the impedances and admittances are referred to the resistance  $R_0 = 1$  by an immittance normalization. It is easy to show then that, numerically, the impedance matrix of the T section is the same as the admittance matrix of the Pi section and vice versa. Hence, under the assumption of Eqs. (187a, b, c) the T and the Pi sections are duals of each other. We also observe that the duality reflects in the elements of the chain matrices: In their numerals the exchange between  $A_T$  and  $D_T$  and the exchange between  $B_T$  and  $C_T$  yields  $A_{Pi}$ ,  $D_{Pi}$ , and  $B_{Pi}$ ,  $C_{Pi}$ .

It is well known that a two-port in the structure of a ladder network that implies only passive  $R$ ,  $L$ , and  $C$  elements always has a dual two-port. Hence, the perfectly coupled and shunt-augmented T — including eventual port-impedances — is also a network of the ladder type, implies  $R$ ,  $L$ ,  $C$  elements, positive or negative, and has a dual two-port. Its dual is a perfectly coupled and series-augmented Pi. This circuit is shown in Figure 22. The branch notations in this figure are admittances. In order to enhance this fact, we added an asterisk to the notations  $u$ ,  $v$ , and  $w$ . Similarly, as in considering the pcsa T, the equation

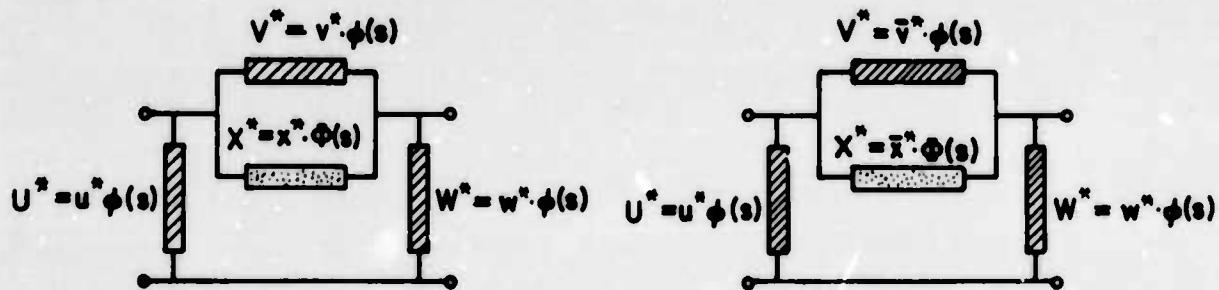
$$1/u^* + 1/v^* + 1/w^* = 0 \quad (191)$$

holds. This equation justifies referring to the term "perfectly coupled". It is evident that, depending on the magnitude of the positive ratio coefficient  $n$ , we can discriminate between a type (A) and a type (B) Pi; when  $n$  is negative  $v^* = -\bar{v}^*$  also is negative resulting in a type (C) Pi. In accordance with the earlier observations, the perfectly coupled part  $v^* \cdot \phi(s)$  of the series branch in the Pi is augmented in parallel by the admittance  $\bar{x}^* \cdot \Phi(s)$ . The elements of the chain matrix of the perfectly coupled and series-augmented Pi are as follows:

$$\hat{A}_{Pi} = w^* + v^* + x^* = \frac{v^*}{n} \phi(s) + x^* \cdot \Phi(s) \quad (192a)$$

$$\hat{B}_{Pi} = 1 \quad (192b)$$

$$\begin{aligned} \hat{C}_{Pi} = & \underbrace{(u^*v^* + u^*w^* + v^*w^*)}_{0} + x^*(u^* + w^*) \\ & + v^*x^* \frac{(n-1)^2}{n} \phi(s) \cdot \Phi(s) \end{aligned} \quad (192c)$$



$$\begin{array}{ll}
 U'' = v'' \cdot (n-1) \geq 0 \text{ if } w'' = v'' \left( \frac{1}{n} - 1 \right) \leq 0 & U'' = \bar{v}'' \cdot (\bar{n}+1) \text{ and } w'' = v'' \left( \frac{1}{\bar{n}} + 1 \right) > 0 \\
 v'' \text{ and } x'' > 0 & \bar{x}'' = x'' < 0 \\
 \text{Type (A)} \rightarrow u'' < 0 & v'' = -\bar{v}'' \text{ and } n = -\bar{n} < 0 \\
 \text{Type (B)} \rightarrow w'' < 0 & \text{Type (C)}
 \end{array}$$

Figure 22. The Perfectly Coupled and Series-Augmented Pi of Types (A) or (B) Left, and of Type (C) Right

$$\hat{D}_{Pi} = U'' + V'' + X'' = v'' \cdot n \cdot \phi(s) , \quad (192d)$$

$$\hat{E}_{Pi} = V'' + X'' = v'' \cdot \phi(s) + x'' \cdot \Phi(s) . \quad (192e)$$

Equations (189a, . . . , e) apply to pcsa Pi's of type (A) or (B); when applied to type (C)  $v''$  is more conveniently replaced by  $-\bar{v}''$  and  $n$  is replaced by  $-\bar{n}$ , as we used to do in the case of a pcsa T of type (C).

We can state the following theorem:

THEOREM 6 (concerning the dual of a pcsa T)

The dual of a pcsa T is a perfectly coupled and series-augmented Pi. Its branches are the dual circuits of the corresponding branches of a pcsa T. It has the same properties as a pcsa T when the well known rules of duality are properly applied.

It is easy to show that all the previous discussions on the pcsa T are likewise true for the perfectly coupled and series-augmented Pi when the rules of duality are correctly applied. In correspondence to Section 2 of this report, series admittances can be transposed from one port to the other over a Pi causing a change of the ratio constant. In correspondence to Section 3, perfectly matched tandems implying perfectly coupled and series-augmented Pi's are equivalent and have an equivalent lattice two-port in accordance with Section 5. Perfect matching

can be obtained in a tandem by transposing series port admittance in accordance with Section 6. In Section 8 we realized the driving-point impedance of a pcsa T of type (B). The driving-point impedance of a pcsa T of type (A) can be realized by realizing the driving-point admittance of a perfectly coupled and **series-augmented**  $P_i$  in which the second shunt admittance is negative.

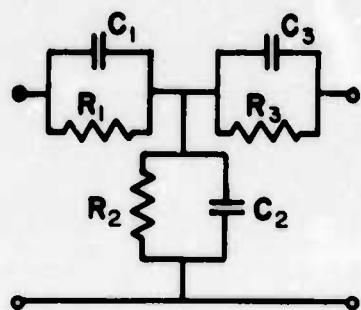
## Part II Numerical Examples

The purpose of several numerical examples presented in this part is to supplement the theory on the perfectly coupled and shunt-augmented T that is presented in Part I. The examples are chosen in such a way that they make the problem as clear as possible. In practical applications one usually has to expect numerical values containing a much higher number of digits. Each example refers to a particular section in Part I. We utilized an earlier report (Haase, 1963) in which formulas are presented by which the elements of admittance functions can immediately be computed and in which these functions for a transformerless realization are cataloged. References to this report are made as Report AFCRL-63-506.

### I. EXAMPLE 1 (referring to Section I)

#### 1.1 Problem

Show that the circuit presented in Figure 23 is a perfectly coupled T.



$$\begin{array}{ll}
 C_1 = 1/6 & R_1 = 10 \\
 C_2 = 1/2 & R_2 = 10/3 \\
 C_3 = -1/1.5 & R_3 = -2.5
 \end{array}$$

Figure 23. Example of a Perfectly Coupled T of Type (B)

### 1.2 Solution

The impedance functions of the branches of the T circuit are of the type

$$Q_2^{-1} = \frac{1}{k(s + a_0^2)}$$

according to Table 4 in Report AFCRL-63-506. According to the same table we find the impedances

$$Z_1 = \frac{6}{s + 6/10} = \frac{6}{s + 0.6} , \quad \text{hence } u = 6$$

$$Z_2 = \frac{2}{s + 6/10} = \frac{2}{s + 0.6} , \quad \text{hence } v = 2$$

$$Z_3 = -\frac{1.5}{s + 1.5/2.5} = \frac{1.5}{s + 0.6} , \quad \text{hence } w = -1.5 .$$

It is essential that  $Z_1$ ,  $Z_2$ , and  $Z_3$  have the same normalized polynomial  $s + 0.6$  in the denominator, so that

$$\phi(s) = 1/(s + 0.6) .$$

It is further necessary that Eq. (3) be true. This is the case since

$$1/u + 1/v + 1/w = 1/6 + 1/2 - 1/1.5 = 0 .$$

This check can also be performed as

$$uv + uw + vw = 12 - 9 - 3 = 0 .$$

We find by Eq. (8a) or by Eq. (8b) that the ratio constant  $n = 4$ . The circuit presented in Figure 23 is a perfectly coupled T of type (B) since  $w$  is negative and  $n$  is positive and greater than 1.

## 2. EXAMPLE 2 (referring to Section 1)

### 2.1 Problem

Show the circuit of a perfectly coupled T of type (C) for which the normalized frequency function is

$$\phi(s) = \frac{s}{s + 0.8}$$

and in which the ratio constant is

$$n = -3$$

and the mutual constant is

$$v = -5.$$

### 2.2 Solution

We introduce according to Eqs. (10a, b)

$$\bar{v} = -v = 5 \quad \text{and} \quad \bar{n} = -n = 3.$$

Then by Eqs. (8a, b)

$$u = \bar{v}(\bar{n} + 1) = 20 \quad w = \bar{v} \left( \frac{1}{\bar{n}} + 1 \right) = \frac{20}{3}.$$

Consequently

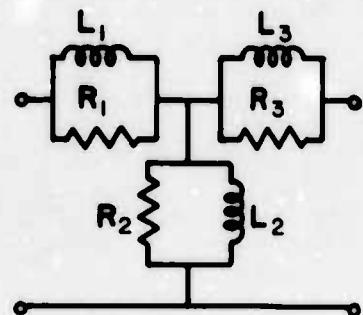
$$U = 18 \frac{s}{s + 0.8},$$

$$V = -5 \frac{s}{s + 0.8},$$

$$W = 4 \frac{s}{s + 0.8}.$$

By Table 1C of Report AFCRL-63-506 we recognize  $\phi(s)$  as being of type  $P_2^{-1}$

By Table 4 of this report we find the circuit and the circuit elements as follows:



$L_1 = 22.5$	$R_1 = 18$
$L_2 = -6.25$	$R_2 = -5$
$L_3 = 5$	$R_3 = 4$

Figure 24. Example of a Perfectly Coupled T of Type (C)

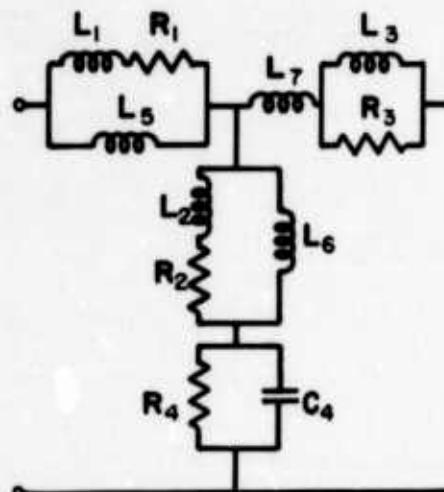
We check the constants of the perfectly coupled T and we find

$$1/u + 1/v + 1/w = 0.05 - 0.2 + 0.15 = 0 \quad \text{o. k.}$$

### 3. EXAMPLE 3 (referring to Section 1)

#### 3.1 Problem

Show that the circuit presented in Figure 25 is a pcsa T. Of what type is it, what are its characterizing constants, and what are the elements of its chain matrix?



$L_1 = 14.4$	$R_1 = 11.52$
$L_2 = 4.8$	$R_2 = 3.84$
$L_3 = -3.75$	$R_3 = -1.125$
$L_4 = \text{none}$	$R_4 = 1/2.4$
$L_5 = 24$	
$L_6 = 8$	
$L_7 = -2.25$	$C_4 = 4$

Figure 25. Example of a pcsa T

## 3.2 Solution

We find by Table 5C of Report AFCRL-63-506 that the circuits U and W of the T are of the same type  $Q_3$ . Applying this table we obtain the impedance functions

$$U = k_1 s \frac{s + a_{11}^2}{s + b_{11}^2} ,$$

where

$$k_1 = \frac{L_1 L_5}{L_1 + L_5} = \frac{345.6}{38.4} = 9 ,$$

$$a_{11}^2 = \frac{R_1}{L_1} = \frac{11.52}{14.4} = 0.8 ,$$

$$b_{11}^2 = \frac{R_1}{L_1 + L_5} = \frac{11.52}{38.4} = 0.3 .$$

We find the impedance function

$$V = k_2 s \frac{s + a_{12}^2}{s + b_{12}^2} ,$$

where

$$k_2 = \frac{L_2 L_6}{L_2 + L_6} = \frac{38.4}{12.8} = 3 ,$$

$$a_{12}^2 = \frac{R_2}{L_2} = 0.8 = a_{11}^2 ,$$

$$b_{12}^2 = \frac{R_2}{L_2 + L_6} = 0.3 = b_{11}^2 .$$

We find the impedance function

$$W = k_3 s \frac{s + a_{13}^2}{s + b_{13}^2} ,$$

where

$$k_3 = L_7 = -2.25 ,$$

$$a_{13}^2 = R_3 \frac{L_3 + L_7}{L_3 L_7} = -1.125 \frac{-6}{8.4375} = \frac{6.75}{8.4375} = 0.8 = a_{12}^2$$

$$b_{13}^2 = \frac{R_3}{L_3} = \frac{-1.125}{3.75} = 0.3 = b_{12}^2 .$$

By Table 4 of Report AFCRL-63-506 we find the impedance function of the shunt-augmentation as

$$X = \frac{1}{k_x(s + a_x^2)} ,$$

where

$$k_x = C_4 = 4 = 1/X$$

and

$$a_x^2 = \frac{1}{R_4 C_4} = \frac{1}{0.6} .$$

From the results obtained so far we find that

$$\phi(s) = s \frac{s + 0.8}{s + 0.3} ,$$

$$\Phi(s) = \frac{1}{s + 1/0.6} ,$$

$$u = k_1 = 9 , \quad v = k_2 = 3 , \quad w = k_3 = -2.25 .$$

We check

$$uv + uw + vw = 27 - 20.25 - 6.75 = 0 \quad \text{o.k.}$$

By Eq. (8a) or Eq. (8b) we find that  $n = 4$ . Hence the pcsa T presented in Figure 25 is of type (B). Its matrix elements are obtained by Eqs. (16a, ..., e) as follows:

$$\hat{A} = 12s \frac{s+0.8}{s+0.3} + \frac{0.25}{s+1/0.6} ,$$

$$\hat{B} = 0.75 \frac{9}{4}s \frac{s+0.8}{s+0.3} \frac{1}{s+1/0.6} ,$$

$$\hat{C} = 1 ,$$

$$\hat{D} = 0.75s \frac{s+0.8}{s+0.3} + \frac{0.25}{s+1/0.6} ,$$

$$\hat{E} = 3s \frac{s+0.8}{s+0.3} + \frac{0.25}{s+1/0.6} .$$

In order to get rid of the numerator polynomials we multiply each element by

$$(s+0.3)(s+1/0.6) = s^2 + 0.59s/0.3 + 0.5$$

and we obtain:

$$\hat{A} = 12s^3 + 29.6s^2 + 16.25s + 0.075 ,$$

$$\hat{B} = 1.6875s(s+0.8) ,$$

$$\hat{C} = s^2 + s \frac{1.18}{0.6} + 0.5 ,$$

$$\hat{D} = 0.75s^3 + 1.85s^2 + 1.25s + 0.075 ,$$

$$\hat{E} = 3s^3 + 7.4s^2 + 4.25s + 0.075 .$$

We will now perform a check whether or not the matrix elements thus obtained are correct. Since the pcsa T is a passive two-port  $AD - BC = 1$  or  $\hat{A}\hat{D} - \hat{B}\hat{C} = \hat{E}^2$  must hold.

For the purpose of this check we have to multiply the polynomial  $\hat{A}$  with the polynomial  $\hat{D}$ , the polynomial  $\hat{B}$  with the polynomial  $\hat{C}$ , and we have to square the polynomial  $\hat{E}$ . This is somewhat cumbersome work to do, especially when the polynomials are of higher degrees, but we cannot avoid doing it. But since the previous check equation must hold for any frequency  $s$ , we will first perform what we refer to as a parity check. We substitute  $s = 1$ , then this evaluation is just the algebraic sum of all the coefficients in the polynomial. In our example

$$\begin{aligned}\hat{A}_{s=1} &= 57.925, & \hat{B}_{s=1} &= 3.0375, & \hat{C}_{s=1} &= 2.08/0.6 \\ \hat{D}_{s=1} &= 3.925, & \hat{E}_{s=1} &= 14.725.\end{aligned}$$

We obtain

$$\begin{aligned}57.925 \times 3.925 - 3.0375 \times 2.08/0.6 &= 227.355625 - 10.53 \\ &= 216.825625\end{aligned}$$

$$14.725^2 = 216.825625.$$

Hence, our result is correct as far as the parity check is concerned. The parity check of course is not as reliable as the substitution of the polynomials themselves. But as to the author's experience in many computations, it is very rare that a computational error is not turned out by the parity check. Throughout these examples we will be satisfied when the parity check finds no errors.

#### 4. EXAMPLE 4 (referring to Section 2)

##### 4.1 Problem

Figure 26 shows a pesa T in which the perfectly coupled branches are inductances and the shunt-augmentation is a resistance. At the in-port we also find a resistance. The figure also shows the technical equivalence using a perfectly coupled transformer. Transpose the in-port resistance to the out-port, and for a check retranspose again.

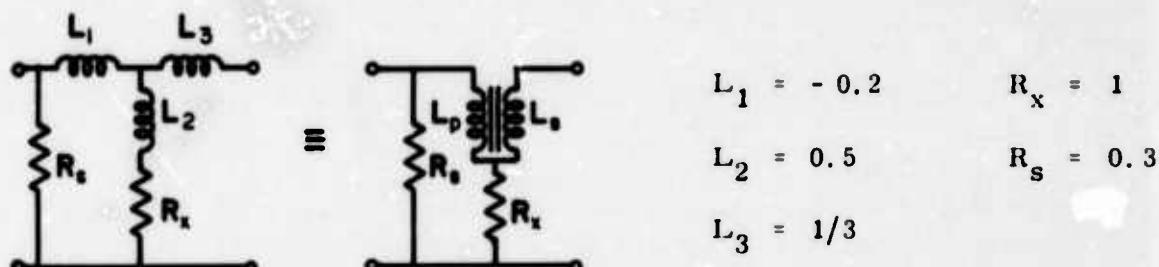


Figure 26. pcsa T With In-Port Impedance

#### 4.2 Solution

Since the elements in the perfectly coupled branches are all inductances, the implied normalized frequency function is

$$\phi(s) = s .$$

The shunt-augmentation is a mere resistance, hence the frequency function implied in the augmentation is

$$\Phi(s) = 1 .$$

We will now express the branches of the pcsa T in the familiar notations:

$$U = u \cdot \phi(s) = -0.2s \longrightarrow u = -0.2$$

$$V = v \cdot \phi(s) = 0.5s \longrightarrow v = 0.5$$

$$W = w \cdot \phi(s) = \frac{s}{3} \longrightarrow w = 1/3 .$$

We check

$$1/u + 1/v + 1/w = -5 + 2 + 3 = 0 , \quad \text{o.k.}$$

By Eq. (8a) we find

$$u = v(n-1) \quad \text{or} \quad -0.2 = 0.5(n-1) .$$

Hence

$$n-1 = -0.4 \quad \text{or} \quad n = 0.6 .$$

The port impedance is a resistance. Hence, it implies the same frequency function  $\Phi(s) = 1$  as the shunt-augmentation and can thus be transposed to the out-port. In our familiar notation

$$X_s = R_s = 0.3, \quad \text{hence} \quad x_s = 0.3.$$

Before we perform the transposition, we will compute the constants of the perfectly coupled transformer in the circuit at the right in Figure 26. By Eqs. (5a, b)

$$L_p = L_1 + L_2 = -0.2 + 0.5 = 0.3,$$

$$L_s = L_2 + L_3 = 0.5 + 1/3 = 5/6.$$

By Eqs. (6a, b), the mutual inductance of the transformer

$$M = \sqrt{L_p L_s} = \sqrt{0.25} = 0.5,$$

the transformer ratio

$$n = \sqrt{L_p/L_s} = \sqrt{0.36} = 0.6.$$

It is necessary that  $M = L_2$  and that the transformer ratio obtained equals the previously computed ratio constant  $n$ .

We now transpose the in-port resistance to the out-port and we obtain the circuits shown in Figure 27.

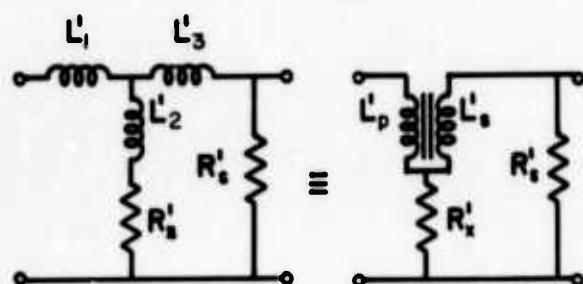


Figure 27. pcsa T With Out-Port Impedance

By Eq. (32)

$$n' = n \frac{\frac{x_s}{x_s + x(n-1)^2}}{s} = 0.6 \frac{0.3}{0.3 + 0.16} = \frac{0.18}{0.46} = 0.391304 .$$

By Eq. (33)

$$\frac{x'}{s} = \frac{\frac{x}{s}}{n n'} = \frac{0.3}{0.6 \times 0.391304} = 1.277781 .$$

By Eq. (34)

$$\begin{aligned} \frac{1}{x'} &= \frac{1}{x} + \frac{1}{x_s} - \frac{1}{x_s} = 1 + 3.333333 - 0.782607 \\ &= 3.550726 = 1/0.281633 , \end{aligned}$$

hence

$$x' = 0.281633 .$$

By Eq. (35)

$$v' = x' \cdot K ,$$

where by Eq. (24)

$$K = \frac{v}{x} = \frac{0.5}{1} = 0.5 .$$

Hence

$$v' = \frac{0.281633}{2} = 0.140817 .$$

With these results the elements in the circuit at the left in Figure 27 are:

$$L'_1 = v' (n' - 1) = 0.140817 \times (-0.608696) = -0.085715 ,$$

$$L'_2 = v' = 0.140817 ,$$

$$L'_3 = v' \left( \frac{1}{n'} - 1 \right) = 0.140817 \times 1.555556 = 0.219049 .$$

Further,

$$R'_x = x' = 0.281633 \quad \text{and} \quad R'_s = x'_s = 1.277781 .$$

We will also check

$$\begin{aligned} 1/u' + 1/v' + 1/w' &= -11.666569 + 7.101415 + 4.565189 \\ &= 0.000035 . \end{aligned}$$

The fact that we do not obtain exactly zero is due to some inaccuracy that is brought in by rounding the last digits in the computation. The deviation, however, can be tolerated.

We are now able to compute the inductances and the transformer ratio in the circuit shown at the right in Figure 27. By Eqs. (5a, b)

$$L'_p = L'_1 + L'_2 = -0.085715 + 0.140817 = 0.055102 ,$$

$$L'_s = L'_2 + L'_3 = 0.140817 + 0.219049 = 0.359866 .$$

Hence

$$M' = \sqrt{L_p L_s} = \sqrt{0.019829} = 0.140815 = L'_2 ,$$

$$n' = \sqrt{L_p/L_s} = \sqrt{0.153118} = 0.391303$$

as it has been obtained previously.

Both circuits shown at the left in Figures 26 and 27 imply pesa T's of type (A) since  $u$  and  $u'$  are negative and  $n$  and  $n'$  are positive and smaller than 1.

Let us now perform the retransposition of  $R'_s$  in Figure 27 to the in-port so that the circuits in Figure 26 are obtained again. By Eq. (25)

$$\begin{aligned} n &= n' + \frac{x'}{x'_s} \frac{(n' - 1)^2}{n'} = 0.391304 + \frac{0.281633}{1.277781} \frac{0.608696^2}{0.391304} \\ &= 0.391304 + 0.220408 \times 0.946862 = 0.6 , \quad \text{o. k.} \end{aligned}$$

By Eq. (28)

$$x_s = x'_s \cdot nn' = 1.277781 \times 0.6 \times 0.391304 = 0.3, \text{ o.k.}$$

By Eq. (29)

$$\begin{aligned} \frac{1}{x} &= \frac{1}{x'} + \frac{1}{x'_s} - \frac{1}{x_s} = 3.550726 + 0.782607 - 3.333333 \\ &= 1, \text{ o.k.} \end{aligned}$$

By Eq. (30)

$$v = x \cdot K = 1 \times 0.5 = 0.5, \text{ o.k.}$$

Thus we obtain the same results as originally given.

By the forward transposition we changed the original ratio coefficient  $n = 0.6$  to the value  $n' = 0.391303$ . For this purpose we transposed the total in-port resistance to the out-port. Evidently, when we transpose only part of the in-port resistance, we could obtain any  $0.6 > n' > 0.391303$ . From a practical point of view it would be, for instance, advantageous for the production of the transformer when we could obtain  $n' = 0.5$ . Let us ask therefore: what is the necessary in-port resistance  $R'_s$  to be transposed over the pesa in Figure 26 in order to obtain a transformer with the ratio 1:2?

By Eq. (32)

$$n' = n \frac{x_s}{x_s + x(n-1)^2}$$

we obtain the answer immediately. We substitute

$$n' = 0.5, n = 0.6, x = 1, \text{ and } x_s = R'_s$$

$$\frac{0.5}{0.6} = \frac{R'_s}{R'_s + 0.16} \text{ with the solution}$$

$$R'_s = 0.8.$$

This resistance  $R'_s$  has to be split off from  $R_s = 0.3$  in parallel. By this split the resistance  $R_1$  obtained from

$$\frac{1}{R_1} = \frac{1}{R_s} - \frac{1}{R''_s} = 3.333333 - 1.25 = 2.083333 = 1/0.48$$

remains at the in-port. The transposed  $R''_s$  appears as  $R'_s$  at the out-port and the constants of the pcos T go over to the primed values. By Eq. (33)

$$R'_s = \frac{R''_s}{n'n'} = \frac{0.8}{0.5 \times 0.6} = \frac{8}{3} .$$

By Eq. (34)

$$\frac{1}{R'_x} = \frac{1}{R_x} + \frac{1}{R''_s} - \frac{1}{R'_s} = 1 + 1.25 - 0.375 = 1.875 = \frac{3}{1.6} ,$$

hence

$$R'_x = \frac{1.6}{3} .$$

By Eq. (35)

$$L'_2 = R'_x \cdot K = \frac{0.8}{3} .$$

The circuits which we obtain by transposing the component  $R''_s = 0.8$  in the forward direction are shown in Figure 28. By Eqs. (8a, b) we obtain the inductances

$$L'_1 = L'_2 \cdot (n' - 1) = \frac{-0.8}{6} = \frac{-0.4}{3} ,$$

$$L'_3 = L'_2 \cdot \left( \frac{1}{n'} - 1 \right) = L'_2 = \frac{0.8}{3} .$$

We check

$$1/L'_1 + 1/L'_2 + 1/L'_3 = \frac{6}{0.8} + \frac{6}{0.8} = 0 , \quad \text{o. k.}$$

The perfectly coupled transformer shown at the right in Figure 28 has the inductances

$$L'_p = L'_1 + L'_2 = \frac{0.4}{3} ,$$

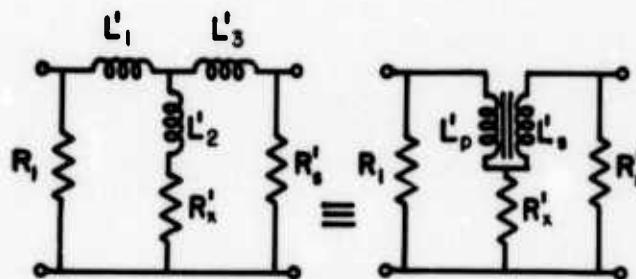
$$L'_s = L'_2 + L'_3 = 2 \cdot L'_2 = \frac{1.6}{3} .$$

The mutual inductance of the transformer is

$$M' = \sqrt{L'_p L'_s} = \sqrt{\frac{0.4 \times 1.6}{9}} = \frac{0.8}{3} = L'_2$$

and the transformer ratio is

$$n' = \sqrt{L'_p / L'_s} = \sqrt{\frac{0.4 \times 3}{1.6 \times 3}} = 0.5 \text{ as expected.}$$



$$\begin{array}{ll} L'_1 = -0.4/3 & R'_s = 8/3 \\ L'_2 = 0.8/3 & R'_x = 1.6/3 \\ L'_3 = 0.8/3 & R_1 = 0.48 \end{array}$$

Figure 28. p.c.s.a T With In-Port and Out-Port Impedance

At this point we would like to perform a parity check on the left side circuits shown in Figures 27 and 28. We substitute  $s = 1$  and we first compute the open-circuit driving-point impedance  $Z_0(1)$  of the circuit at the left in Figure 26. It is

$$Z_0(1) = \frac{1}{\frac{1}{R_s} + \frac{1}{L_1 + L_2 + R_x}} .$$

In this formula the advantage of substituting  $s = 1$ , which is not a physically realizable frequency, becomes evident: the check implies the magnitude of the impedances, but it avoids complex numbers which would occur when a physical frequency  $s = j$  would be used for the substitution.

With the values enumerated for Figure 26 we obtain

$$\begin{aligned} Z_0(1) &= \frac{1}{3.333333 + 1/(-0.2 + 0.5 + 1)} = \frac{1}{3.333333 + 0.769231} \\ &= \frac{1}{4.102564} = 0.243750 . \end{aligned}$$

The driving-point impedance does not, however, include the element  $L_3$  of the circuit shown at the left in Figure 26. Therefore we also compute the driving-point impedance  $Z_s(1)$  that is measured when the secondary terminals of the circuit are shorted.

$$\begin{aligned}
 Z_s(1) &= \frac{1}{\frac{1}{R_s} + \frac{1}{L_1 + \frac{1}{\frac{1}{L_2 + R_x} + \frac{1}{L_3}}}} \\
 &= \frac{1}{3.33333 + \frac{1}{-0.2 + \frac{1}{1/(0.5 + 1) + 3}}} \\
 &= \frac{1}{3.33333 + \frac{1}{-0.2 + \frac{1}{3.66666}}} \\
 &= \frac{1}{3.33333 + \frac{1}{0.072727}} = \frac{1}{17.083384} = 0.058536 .
 \end{aligned}$$

The driving-point impedance  $Z_0(1)$  of the circuit shown at the left in Figure 27 is

$$\begin{aligned}
 Z_0(1) &= L'_1 + \frac{1}{\frac{1}{L'_2 + R'_x} + \frac{1}{L'_3 + R'_s}} \\
 &= -0.085715 + \frac{1}{1/(0.140817 + 0.281633) + 1/(0.219049 + 1.277781)} \\
 &= -0.085715 + \frac{1}{2.367144 + 0.668079} \\
 &= -0.085715 + 0.329465 = 0.243750 , \text{ o.k.}
 \end{aligned}$$

The driving-point impedance  $Z_s(1)$  of the shorted circuit shown at the left in Figure 27 is

$$\begin{aligned}
 Z_s(1) &= L'_1 + \frac{1}{\frac{1}{L'_2 + R'_x} + \frac{1}{L'_3}} \\
 &= -0.085715 + \frac{1}{1/(0.140817 + 0.281633) + 1/0.219049}
 \end{aligned}$$

(equation continued)

$$Z_s(1) = -0.85715 + \frac{1}{2.367144 + 4.565189} \\ = -0.85715 + 0.144252 = 0.058537, \text{ o.k.}$$

By the same formulas we check the circuit shown at the left in Figure 28. The open circuit driving-point impedance  $Z'_0(1)$  that is in parallel to  $R_1$  is

$$Z'_0(1) = -\frac{0.4}{3} + \frac{1}{\frac{3}{0.8 + 1.6} + \frac{3}{0.8 + 8}} \\ = \frac{1}{3} \left( -0.4 + \frac{1}{1/2.4 + 1/8.8} \right) \\ = \frac{1}{3} \left( -0.4 + \frac{2.4 \times 8.8}{11.2} \right) \\ = \frac{1}{3} (-0.4 + 1.885714) = 0.495238.$$

Hence

$$Z_0(1) = \frac{Z'_0(1) \cdot R_1}{Z'_0(1) + R_1} = \frac{0.237715}{0.975238} = 0.243751, \text{ o.k.}$$

The short circuit driving-point impedance  $Z'_s(1)$  that is in parallel with  $R_1$  is

$$Z'_s(1) = -\frac{0.4}{3} + \frac{1}{\frac{3}{0.8 + 1.6} + \frac{3}{0.8}} \\ = \frac{1}{3} \left( -0.4 + \frac{1}{1/2.4 + 1/0.8} \right) \\ = \frac{1}{3} \left( -0.4 + \frac{2.4 \times 0.8}{3.2} \right) = \frac{0.2}{3}.$$

Hence

$$Z_s(1) = \frac{Z'_s(1) \cdot R_1}{Z'_s(1) + R_1} = \frac{0.096}{1.64} = 0.058537, \text{ o.k.}$$

### 5. EXAMPLE 5 (referring to Section 3)

#### 5.1 Problem

Assume that the tandem shown in Figure 7 is given by the constants

$$\begin{array}{ll} v_a = 3, & \bar{v}_b = 2, \\ n_a = 8, & \bar{n}_b = 2.5, \\ x_a = 5, & \bar{x}_b = 10/3. \end{array}$$

Let the normalized frequency functions be

$$\phi(s) = \frac{s + 1.2}{s + 0.4} \quad \text{and} \quad (s) = \frac{1}{s + 0.6}.$$

Show that the tandem is a matched one. Give the elements of its chain matrix and its circuitry.

#### 5.2 Solution

According to the definition of a matched tandem, Eq. (47) must hold. In fact

$$\frac{v_a}{x_a} = 3/5 = 0.6 \quad \text{and} \quad \frac{\bar{v}_b}{\bar{x}_b} = 6/10 = 0.6$$

are the same. Thus the tandem is matched and the constant

$$K = 0.6.$$

According to Eq. (48) the constant

$$k_a = 3/2 = 15/10 = 1.5.$$

By substitution in Eqs. (54a, . . . , e) we obtain the following elements of the chain matrix:

$$\begin{aligned} \hat{A}_{(AB)C} &= 20 \phi^2(s) + \frac{1}{0.36} \Phi^2(s) + \frac{1}{0.6} \left| 10.5 + \frac{49 \times 1.5}{8} \right| \phi(s) \cdot \Phi(s) \\ &= 20 \phi^2(s) + 2.777778 \cdot \Phi^2(s) + 32.8125 \cdot \phi(s) \cdot \Phi(s). \end{aligned}$$

$$\begin{aligned}
 \hat{B}_{(AB)C} &= 5 \left[ \left( 8 \frac{3.5^2}{2.5 \times 1.5} + \frac{7^2}{8 \times 2.5} \right) \phi(s) + \frac{1}{0.6} \left( \frac{3.5^2}{1.5 \times 2.5} + \frac{7^2}{8} \right) \phi(s) \right] \phi(s) \cdot \Phi(s) \\
 &= 5 \left[ 28.583333 \cdot \phi(s) + 15.652778 \cdot \Phi(s) \right] \phi(s) \cdot \Phi(s) , \\
 \hat{C}_{(AB)C} &= \frac{1}{3} \left[ \left( 2.5 + \frac{1.5}{8} \right) \phi(s) + \frac{2.5}{0.6} \Phi(s) \right] \\
 &= \frac{1}{3} \left[ 2.6875 \cdot \phi(s) + 4.166667 \cdot \Phi(s) \right] , \\
 \hat{D}_{(AB)C} &= \frac{1}{20} \phi^2(s) + \frac{1}{0.36} \Phi^2(s) + \frac{1}{0.6} \left[ 0.125 + 0.4 + \frac{3.5^2}{2.5 \times 1.5} \right] \phi(s) \cdot \Phi(s) \\
 &= 0.05 \cdot \phi^2(s) + 2.777778 \cdot \Phi^2(s) + 6.319444 \cdot \phi(s) \cdot \Phi(s) , \\
 \hat{E}_{(AB)C} &= 2.777778 \cdot \Phi^2(s) - \phi^2(s) .
 \end{aligned}$$

Next, we perform a parity check of these results. For this purpose we compute

$$\begin{aligned}
 \phi(1) &= 2.2/1.4 = 1.571429 , & \phi^2(1) &= 2.469389 , \\
 \Phi(1) &= 1/1.6 = 0.625 , & \Phi^2(1) &= 0.390625 .
 \end{aligned}$$

For  $s = 1$  we thus obtain:

$$\begin{aligned}
 \hat{A}_{(AB)C}(1) &= 49.387780 + 1.085069 + 32.226571 = 82.699420 , \\
 \hat{B}_{(AB)C}(1) &= 5 \left[ 44.916678 + 9.782986 \right] 0.982143 = 268.614460 , \\
 \hat{C}_{(AB)C}(1) &= \frac{1}{3} \left[ 4.223215 + 2.604167 \right] = 2.275794 , \\
 \hat{D}_{(AB)C}(1) &= 0.123469 + 1.085069 + 6.206599 = 7.415137 , \\
 \hat{E}_{(AB)C}(1) &= 1.085069 - 2.469389 = -1.384320 .
 \end{aligned}$$

Then

$$\begin{aligned}
 \hat{A} \hat{D} - \hat{B} \hat{C} &= 613.277529 - 611.311176 = 1.916353 , \\
 \hat{E}^2 &= 1.916342 .
 \end{aligned}$$

Thus the parity check proves the results to be acceptable.

The constants of the perfectly coupled branches in the tandem are:

$$u_a = v_a (n_a - 1) = 3 \cdot 7 = 21 = 1/0.047619$$

$$w_a = v_a \left( \frac{1}{n_a} - 1 \right) = -3 \cdot 0.875 = -2.625 = -1/0.380952$$

$$\text{Check: } 1/u_a + 1/v_a + 1/w_a = 0.047619 + 0.333333 - 0.380952 = 0$$

$$u_b = \bar{v}_b (\bar{n}_b + 1) = 2 \cdot 3.5 = 7 = 1/0.142857 ,$$

$$w_b = \bar{v}_b \left( \frac{1}{\bar{n}_b} + 1 \right) = 2 \cdot 1.4 = 2.8 = 1/0.357143 ,$$

$$\text{Check: } 1/u_b + 1/\bar{v}_b + 1/w_b = 0.142857 - 0.5 + 0.357143 = 0 .$$

According to Table 1B of Report AFCRL-63-506 we recognize the function  $\phi(s)$  as being of type  $P_3$

$$\phi(s) = k \frac{s + a_1^2}{s + b_1^2} = \frac{s + 1.2}{s + 0.4}$$

$$\text{so that } a_1^2 = 1.2 \text{ and } b_1^2 = 0.4 , \text{ and } k = 1 .$$

We compute the elements of an impedance function that implies  $\phi(s)$ . Whenever the constant is different from  $k = 1$ , we only have to multiply the inductances and resistances of the circuit by this constant, and we have to divide the capacitances by it. According to Table 9 the circuit shown in Figure 29 has the impedance function  $\phi(s)$  .

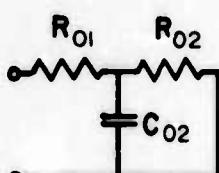


Figure 29. Realization of the Impedance Function  $\phi(s)$

$$R_{01} = 1 ,$$

$$R_{02} = 1 \frac{a_1^2 - b_1^2}{b_1^2} = 0.8/0.4 = 2 ,$$

$$C_{02} = \frac{1}{1} \frac{1}{a_1^2 - b_1^2} = 1/0.8 = 1.25 .$$

The frequency function  $\Phi(s)$  can be recognized as being of type  $Q_2^{-1}$  by Table 1B of Report AFCRL-63-506.

$$\Phi(s) = \frac{1}{k(s + a_0^2)} = \frac{1}{s + 0.6}$$

$$\text{so that } k = 1 \text{ and } a_0^2 = 0.6.$$

According to Table 8 of the report we find that the circuit shown in Figure 30 has the impedance function  $\Phi(s)$ .

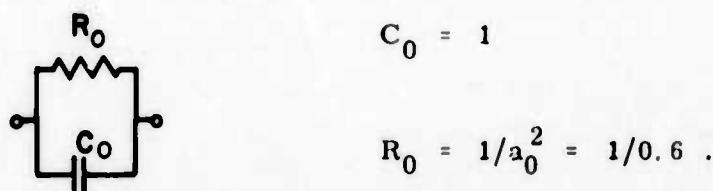


Figure 30. Realization of the Impedance Function  $\Phi(s)$

The circuit of the tandem is shown in Figure 31.

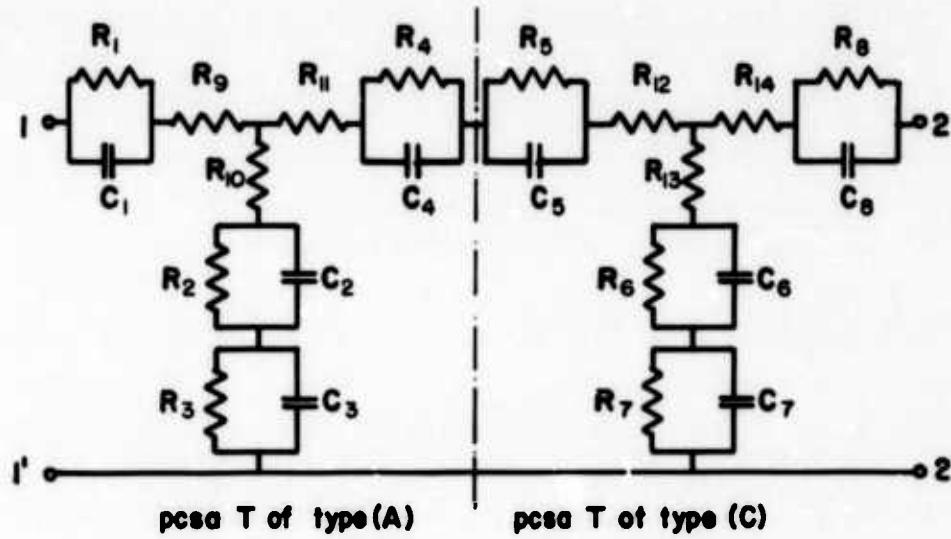


Figure 31. Circuit of the Problem in Example 5

The circuit elements in Figure 31 are as follows:

$$\begin{array}{ll}
 R_1 = u_a \cdot R_{02} = 0.095238 & C_1 = C_{02}/u_a = 26.250026 \\
 R_2 = v_a \cdot R_{02} = 6 & C_2 = C_{02}/v_a = 0.416667 \\
 R_3 = x_a \cdot R_0 = 5/0.6 & C_3 = C_0/x_a = 0.2 \\
 R_4 = w_a \cdot R_{02} = -5.25 & C_4 = C_{02}/w_a = -0.476190 \\
 R_5 = u_b \cdot R_{02} = 14 & C_5 = C_{02}/u_b = 0.178571 \\
 R_6 = -\bar{v}_b \cdot R_{02} = -4 & C_6 = -C_{02}/\bar{v}_b = -0.625 \\
 R_7 = \bar{x}_b \cdot R_0 = 10/1.8 & C_7 = C_0/\bar{x}_b = 0.3 \\
 R_8 = w_b \cdot R_{02} = 5.6 & C_8 = C_{02}/w_b = 0.446429 \\
 R_9 = u_a \cdot R_{01} = 0.047619 & \\
 R_{10} = v_a \cdot R_{01} = 3 & \\
 R_{11} = w_a \cdot R_{01} = -2.625 & \\
 R_{12} = u_b \cdot R_{01} = 7 & \\
 R_{13} = -\bar{v}_b \cdot R_{01} = -2 & \\
 R_{14} = w_b \cdot R_{01} = 2.8 &
 \end{array}$$

We now perform a parity check on our result. The impedance of the circuit in Figure 29, when we substitute  $s = 1$  is

$$Z_{00}(1) = R_{01} + \frac{1}{\frac{1}{R_{02}} + C_{02}} = 1 + \frac{1}{0.5 + 1.25} = 1.571429.$$

For the same frequency parameter  $s = 1$  the impedance of the circuit in Figure 30 is

$$Z_0(1) = \frac{1}{\frac{1}{R_0} + C_0} = \frac{1}{0.6 + 1} = 0.625.$$

A block diagram of the circuit in Figure 31 is shown in Figure 32. All impedances  $X_1, \dots, X_6$  are evaluated for  $s = 1$ .

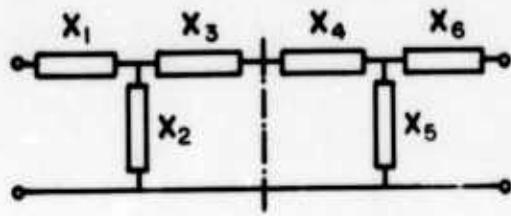


Figure 32. Block Diagram of the Circuit in Figure 31

$$\begin{aligned}
 X_1 &= u_a \cdot Z_{00}(1) = 33 \\
 X_2 &= v_a \cdot Z_{00}(1) + x_a \cdot Z_0(1) = 7.839287 \\
 &\quad = 1/0.127563 \\
 X_3 &= w_a \cdot Z_{00}(1) = -4.125 \\
 X_4 &= u_b \cdot Z_{00}(1) = 11 \\
 X_5 &= -\bar{v}_b \cdot Z_{00}(1) + \bar{x}_b \cdot Z_0(1) = -1.059525 \\
 &\quad = -1/0.943819 \\
 X_6 &= w_b \cdot Z_{00}(1) = 4.4
 \end{aligned}$$

The circuit in Figure 32 is a tandem of two T structures. The first of them has the chain matrix

$$\begin{vmatrix} 1 + X_1/X_2 & X_1 + X_3 + X_1 X_3/X_2 \\ 1/X_2 & 1 + X_3/X_2 \end{vmatrix} = \begin{vmatrix} 5.209579 & 11.510487 \\ 0.127563 & 0.473803 \end{vmatrix}$$

The second T structure has the chain matrix

$$\begin{vmatrix} 1 + X_4/X_5 & X_4 + X_6 + X_4 X_6/X_5 \\ 1/X_5 & 1 + X_6/X_6 \end{vmatrix} = \begin{vmatrix} 9.382009 & 30.280840 \\ 0.943819 & 3.152804 \end{vmatrix}$$

Therefore the circuit shown in Figure 32 in its block diagram has the chain matrix

$$\begin{aligned}
 & \begin{vmatrix} 5.209579 & 11.510487 \\ 0.127563 & 0.473803 \end{vmatrix} \cdot \begin{vmatrix} 9.382009 & 30.280840 \\ 0.943819 & 3.152804 \end{vmatrix} \\
 &= \begin{vmatrix} 59.740133 & 194.040738 \\ 1.643981 & 5.356523 \end{vmatrix}
 \end{aligned}$$

The evaluation of the chain matrix which we found previously was

$$\begin{aligned}
 -\frac{1}{\hat{E}} \begin{vmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{vmatrix} &= -\frac{1}{1.384320} \begin{vmatrix} 82.699420 & 268.614460 \\ 2.275794 & 7.415137 \end{vmatrix} \\
 &= \begin{vmatrix} 59.740103 & 194.040728 \\ 1.643980 & 5.356519 \end{vmatrix} .
 \end{aligned}$$

Both matrices are in good agreement thus validating the parity check.

We want to emphasize that the circuit in Figure 31 shows more circuit elements than necessary. Evidently since  $R_4C_4 = R_5C_5$ , these two parallel circuits can be combined to make one. For the same token the resistances  $R_{11}$  and  $R_{12}$  can be combined to make one resistance. Then, however, the two-port would hardly be recognized as a tandem of two pcsa T's. For this reason, in this particular circuit, as well as in all the circuits of the examples in this report, we do not care about reductions and transformations which intend to make the circuits simpler.

## 6. EXAMPLE 6 (referring to Section 3)

### 6.1 Problem

Assume that the tandem shown in Figure 7 is given by the constants:

$$\begin{aligned}
 v_a &= 5, & \bar{v}_b &= 1, \\
 n_a &= 2.5, & \bar{n}_b &= 0.4, \\
 x_a &= 4, & \bar{x}_b &= 0.8.
 \end{aligned}$$

Let the normalized frequency functions be

$$\phi(s) = (s + 0.2) \frac{s + 1.5}{s + 0.8} \quad \text{and} \quad \Phi(s) = \frac{s + 0.9}{s + 0.4} .$$

Show that the tandem is a perfectly matched one. Show its circuit and its chain matrix. What is the equivalent tandem in accordance to Figure 8?

### 6.2 Solution

Since  $n_a = 1/\bar{n}_b$  the tandem is a perfectly matched one. According to Eq. (55)

$$n_0 = n_a = 1/\bar{n}_b = 2.5 .$$

By  $n_a > 1$  the first section is of type (A). The constants  $K$  and  $k_a$  characterizing the tandem are, according to Eqs. (47) and (48):

$$K = v_a/x_a = \bar{v}_b/\bar{x}_b = 5/4 = 1.25, \quad K^2 = 1.5625 = 1/0.64$$

$$k_a = v_a/\bar{v}_b = x_a/\bar{x}_b = 5.$$

The elements of the chain matrix of the tandem in Figure 7 are obtained by Eqs. (56a, ..., e) as follows:

$$\hat{A}_{AC} = \phi^2(s) + 0.64 \cdot \Phi^2(s) + \frac{1}{3.125} \cdot 18.5 \cdot \phi(s) \cdot \Phi(s)$$

$$= \phi^2(s) + 0.64 \cdot \Phi^2(s) + 5.92 \cdot \phi(s) \cdot \Phi(s),$$

$$\hat{B}_{AC} = \frac{5}{6.25} \cdot 23.5 \left[ \phi(s) + \frac{1}{3.125} \Phi(s) \right] \cdot \phi(s) \cdot \Phi(s)$$

$$= 18.8 \left[ \phi(s) + 0.32 \cdot \Phi(s) \right] \cdot \phi(s) \cdot \Phi(s)$$

$$\hat{C}_{AC} = \frac{6}{5} \left[ \frac{1}{2.5} \phi(s) + \frac{1}{1.25} \Phi(s) \right]$$

$$= 1.2 \left[ 0.4 \phi(s) + 0.8 \Phi(s) \right],$$

$$\hat{D}_{AC} = \phi^2(s) + 0.64 \Phi^2(s) + \frac{1}{3.125} \cdot 9.7 \cdot \phi(s) \cdot \Phi(s)$$

$$= \phi^2(s) + 0.64 \Phi^2(s) + 3.104 \cdot \phi(s) \cdot \Phi(s),$$

$$\hat{E}_{AC} = 0.64 \Phi^2(s) - \phi^2(s).$$

We evaluate

$$\phi(1) = 1.2 \frac{2.5}{1.8} = 1.666667, \quad \Phi(1) = \frac{1.9}{1.4} = 1.357143.$$

The evaluations of the matrix elements for  $s = 1$  thus are:

$$\hat{A}_{AC}(1) = 2.777778 + 1.178776 + 13.390478 = 17.347032,$$

$$\hat{B}_{AC}(1) = 18.8 \times 2.100953 \times 2.261905 = 89.340533,$$

$$\hat{C}_{AC}(1) = 1.2 \times 1.752381 = 2.102857,$$

$$\hat{D}_{AC}(1) = 2.777778 + 1.178776 + 7.020953 = 10.977507$$

$$\hat{E}_{AC}(1) = -1.599003 .$$

We check:  $17.347032 \times 10.977507 - 89.340533 \times 2.102857 = 2.556800$

$$1.599003^2 = 2.556811 , \text{ o.k.}$$

The frequency function  $\phi(s)$  is recognized as a function of the type  $Q_4$  by Table 1C of Report AFCRL-63-506.

$$Q_4 = k (s + a_0^2) \frac{(s + a_1^2)}{(s + b_1^2)} = (s + 0.2) \frac{s + 1.5}{s + 0.8} .$$

By comparison:  $k = 1$ ,  $a_0^2 = 0.2$ ,  $a_1^2 = 1.5$ ,  $b_1^2 = 0.8$  .

A realization of the impedance  $\phi(s)$  is shown in Figure 33 as it is obtained by Table 11 of the report.

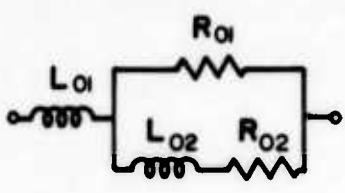


Figure 33. Realization of the Impedance Function  $\phi(s)$

Auxiliary Constants:

$$a^4 = b_1^2(a_0^2 + a_1^2) - (b_1^4 + a_0^2 a_1^2)$$

$$\rho = 0.8 \times 1.7 - 0.94 = 0.42 ,$$

$$= \frac{a_0^2 a_1^2}{a^4} = \frac{0.3}{0.42} = 0.714286 .$$

$$L_{01} = k = 1 ,$$

$$R_{01} = k \frac{a^4(1 + \rho)}{b_1^2} = 0.9 ,$$

$$L_{02} = k \frac{a^4(1 + \rho)^2}{b_1^4} = 1.928572 ,$$

$$R_{02} = k \frac{a^4(1 + \rho) \cdot \rho}{b_1^2} = 0.642858 .$$

The frequency function  $\Phi(s)$  is recognized as a function of type  $P_3$  by Table 1B of Report AFCRL-63-506.

$$P_3 = k \frac{s + a_1^2}{s + b_1^2} = \frac{s + 0.9}{s + 0.4} .$$

By comparison:  $k = 1$ ,  $a_1^2 = 0.9$ ,  $b_1^2 = 0.4$ .

A realization of the impedance function  $\Phi(s)$  is shown in Figure 34 as it is obtained by Table 9 of the report.

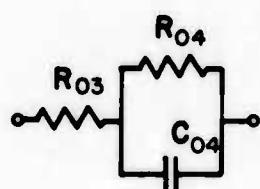


Figure 34. Realization of the Impedance Function  $\Phi(s)$

$$R_{03} = k = 1 ,$$

$$R_{04} = k \frac{a_1^2 - b_1^2}{b_1^2} = \frac{0.5}{0.4} = 1.25 ,$$

$$C_{04} = \frac{1}{k} \frac{1}{a_1^2 - b_1^2} = 2 .$$

The realization of the complete tandem is shown in Figure 35. The pcsa T at the left end is of type (A) and has the constants  $v_a = 5$ ,  $x_a = 4$ , and  $n_a = 2.5$ . Hence,

$$u_a = v_a(n - 1) = 7.5 ,$$

$$w_a = v_a \left( \frac{1}{n_a} - 1 \right) = -3 .$$

The pcsa T at the right end is of type (C) and has the constants  $\bar{v}_b = 1$ ,  $\bar{x}_b = 0.8$ ,  $\bar{n}_b = 0.4$ . Hence,

$$u_b = \bar{v}_b(\bar{n}_b + 1) = 1.4 ,$$

$$w_b = \bar{v}_b \left( \frac{1}{\bar{n}_b} + 1 \right) = 3.5 .$$

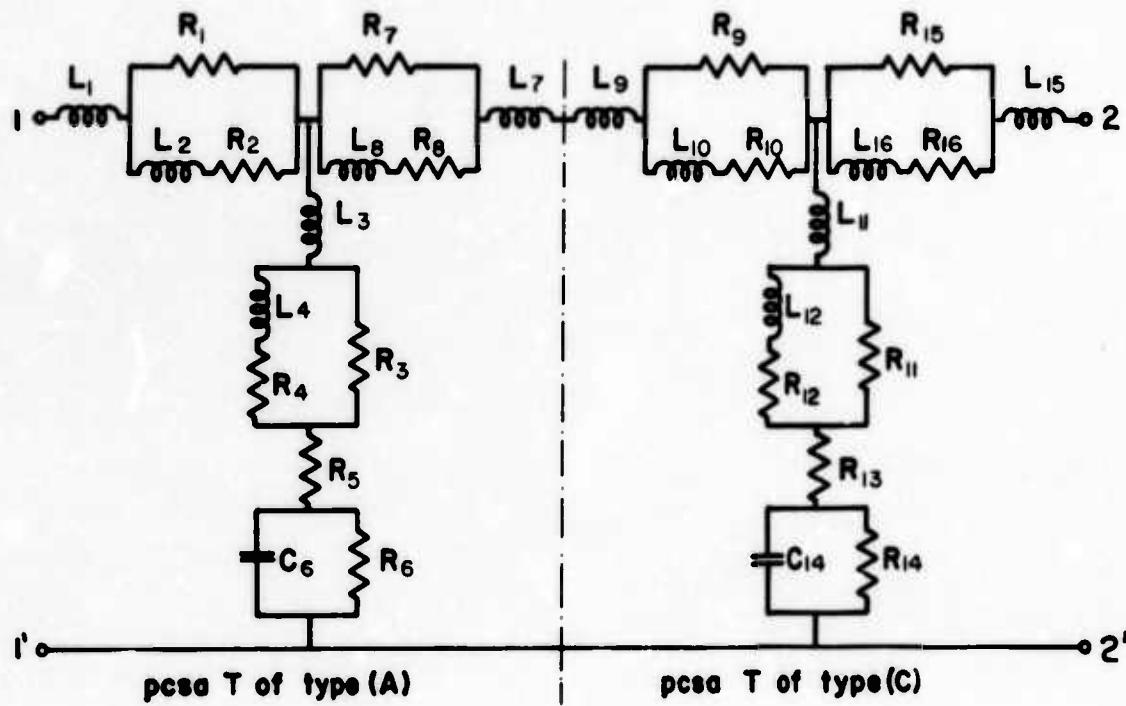


Figure 35. Tandem Computed in Example 6

$$L_1 = u_a L_{01} = 7.5$$

$$L_2 = u_a L_{02} = 14.464290$$

$$L_3 = v_a L_{01} = 5$$

$$L_4 = v_a L_{02} = 9.642860$$

$$C_6 = C_{04}/x_a = 0.5$$

$$L_7 = w_a L_{01} = -3$$

$$L_8 = w_a L_{02} = -5.785716$$

$$L_9 = u_b L_{01} = 1.4$$

$$L_{10} = u_b L_{02} = 2.7$$

$$L_{11} = v_b L_{01} = -1$$

$$L_{12} = v_b L_{02} = -1.928572$$

$$C_{14} = C_{04}/\bar{x}_b = 2.5$$

$$L_{15} = w_b L_{01} = 3.5$$

$$L_{16} = w_b L_{02} = 6.75$$

$$R_1 = u_a R_{01} = 6.75$$

$$R_2 = u_a R_{02} = 4.821435$$

$$R_3 = v_a R_{01} = 4.5$$

$$R_4 = v_a R_{02} = 3.214290$$

$$R_5 = x_a R_{03} = 4$$

$$R_6 = x_a R_{04} = 5$$

$$R_7 = w_a R_{01} = -2.7$$

$$R_8 = w_a R_{02} = -2.142860$$

$$R_9 = u_b R_{01} = 1.26$$

$$R_{10} = u_b R_{02} = 0.9$$

$$R_{11} = v_b R_{01} = -0.9$$

$$R_{12} = v_b R_{02} = -0.642858$$

$$R_{13} = \bar{x}_b R_{03} = 0.8$$

$$R_{14} = \bar{x}_b R_{04} = 1$$

$$R_{15} = w_b R_{01} = 3.15$$

$$R_{16} = w_b R_{02} = 2.25$$

We will now determine the equivalent tandem. For this purpose we first need the constants  $k_b$  and  $v_d$ . They are obtained by Eqs. (67) and (68). From Eq. (67) we obtain

$$k_b = \left[ \frac{n_0 + 1}{n_0 - 1} \right]^2 \frac{1}{k_a} = \frac{3.5^2}{1.5^2} \frac{1}{5} = \frac{12.25}{11.25} = 1.088889 ,$$

and from Eq. (68)

$$v_d = \frac{(1 + k_b)v_a}{1 + k_a} = \frac{2.088889 \times 5}{6} = 1.740740 .$$

The tandem that is shown in its block diagram in Figure 8 is characterized by the following constants:

$$v_d = 1.740740 ,$$

$$n_d = 1/n_0 = 0.4 ,$$

$$u_d = v_d(n_d - 1) = -1.044444 ,$$

$$w_d = v_d \left( \frac{1}{n_d} - 1 \right) = 2.611111 ,$$

$$x_d = v_d/K = 1.393393 \text{ as follows from Eq. (70)} .$$

By Eq. (58)

$$\bar{v}_c = v_d/k_b = 1.598639 ,$$

and by Eq. (57)

$$\bar{x}_c = \bar{v}_c/K = 1.278911 .$$

The ratio coefficient

$$\bar{n}_c = 1/n_d = n_0 = 2.5 .$$

Hence,

$$u_c = \bar{v}_c(\bar{n}_c + 1) = 5.595237 ,$$

$$w_c = \bar{v}_c \left( \frac{1}{\bar{n}_c} + 1 \right) = 2.238095 .$$

The matrix elements of the second tandem can now be computed by Eqs. (66a, ..., e). We obtain:

$$\begin{aligned}\hat{A}_{CB} &= \phi^2(s) + 0.64 \Phi^2(s) + \frac{1}{3.125} 18.5 \phi(s) \cdot \Phi(s) \\ &= \phi^2(s) + 0.64 \Phi^2(s) + 5.92 \phi(s) \cdot \Phi(s) ,\end{aligned}$$

$$\begin{aligned}\hat{B}_{CB} &= \frac{1.740740}{1.25} 13.5 \left[ \phi(s) + \frac{1}{3.125} \Phi(s) \right] \phi(s) \cdot \Phi(s) \\ &= 18.8 \left[ \phi(s) + 0.32 \Phi(s) \right] \phi(s) \cdot \Phi(s) ,\end{aligned}$$

$$\begin{aligned}\hat{C}_{CB} &= \frac{2.088889}{1.740740} \left[ 0.4 \phi(s) + 0.8 \Phi(s) \right] \\ &= 1.2 \left[ 0.4 \phi(s) + 0.8 \Phi(s) \right] ,\end{aligned}$$

$$\begin{aligned}\hat{D}_{CB} &= \phi^2(s) + 0.64 \Phi^2(s) + \frac{1}{3.125} 9.7 \phi(s) \cdot \Phi(s) \\ &= \phi^2(s) + 0.64 \Phi^2(s) + 3.104 \phi(s) \cdot \Phi(s) ,\end{aligned}$$

$$\hat{E}_{CB} = 0.64 \Phi^2(s) - \phi^2(s) .$$

As we expected, the elements  $\hat{A}_{CB}, \dots, \hat{E}_{CB}$  are the same as the previously obtained elements  $\hat{A}_{AC}, \dots, \hat{E}_{AC}$ .

The circuit of the tandem that is equivalent to the tandem shown in Figure 35 is presented in Figure 36. The magnitudes of the circuit elements are obtained by multiplying the respective elements of the circuits shown in Figures 33 and 34 by the constants of the first and the second pesa T.

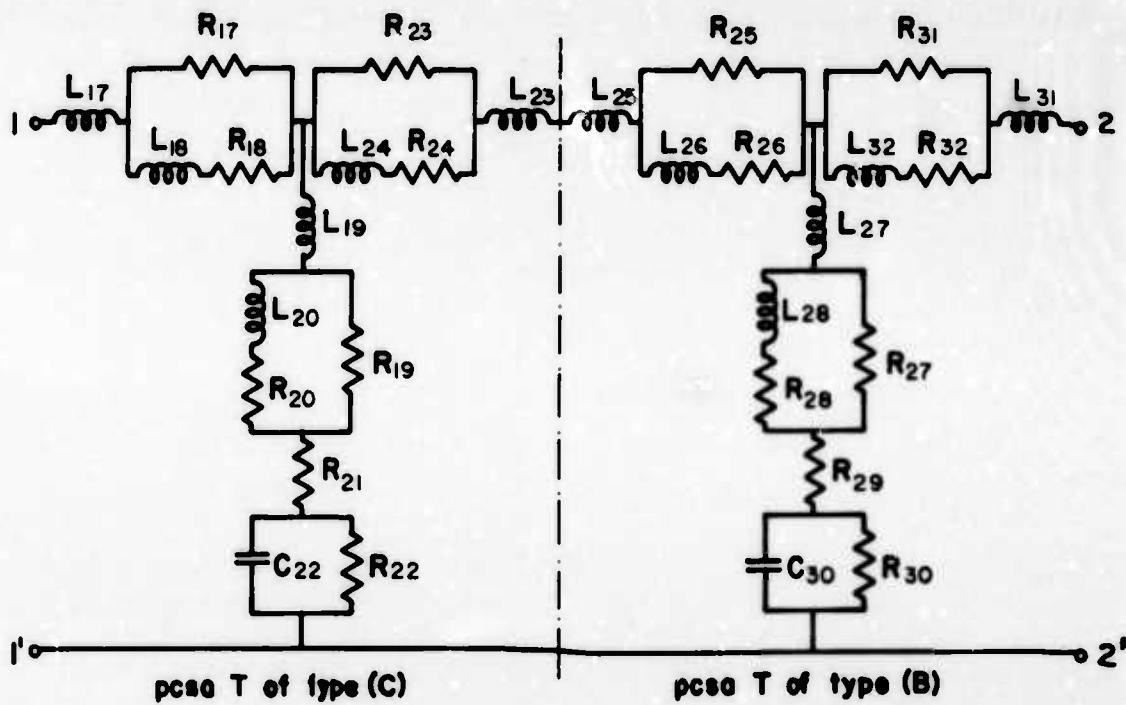


Figure 36. Tandem Equivalent to That in Figure 35

$L_{17} = u_c L_{01} = 5.595237$	$R_{17} = u_c R_{01} = 5.035713$
$L_{18} = u_c L_{02} = 10.790817$	$R_{18} = u_c R_{02} = 3.596943$
$L_{19} = v_c L_{01} = -1.598639$	$R_{19} = v_c R_{01} = -1.438775$
$L_{20} = v_c L_{02} = -3.083090$	$R_{20} = v_c R_{02} = -1.027698$
	$R_{21} = \bar{x}_c R_{03} = 1.278911$
$C_{22} = C_{04}/\bar{x}_c = 1.563830$	$R_{22} = \bar{x}_c R_{04} = 1.598639$
$L_{23} = w_c L_{01} = 2.238095$	$R_{23} = w_c R_{01} = 2.014286$
$L_{24} = w_c L_{02} = 4.316327$	$R_{24} = w_c R_{02} = 1.438777$
$L_{25} = u_d L_{01} = -1.044444$	$R_{25} = u_d R_{01} = -0.94$
$L_{26} = u_d L_{02} = -2.014285$	$R_{26} = w_d R_{02} = -0.671429$
$L_{27} = v_d L_{01} = 1.740740$	$R_{27} = v_d R_{01} = 1.566667$
$L_{28} = v_d L_{02} = 3.357142$	$R_{28} = v_d R_{02} = 1.119048$
	$R_{29} = x_d R_{03} = 1.393393$
$C_{30} = C_{04}/x_d = 1.435345$	$R_{30} = x_d R_{04} = 1.741741$
$L_{31} = w_d L_{01} = 2.611111$	$R_{31} = w_d R_{01} = 2.35$
$L_{32} = w_d L_{02} = 5.035716$	$R_{32} = w_d R_{02} = 1.678574$

### 7. EXAMPLE 7 (referring to Section 5)

#### 7.1 Problem

What is the lattice equivalence of the perfectly matched tandems obtained in the previous Example 6?

#### 7.2 Solution

The tandem shown in Figure 35 is characterized by the constants

$$\begin{aligned} v_a &= 5, & \bar{v}_b &= 1 \\ n_a &= 2.5 & \bar{n}_b &= 0.4 = 1/2.5, \\ x_a &= 4, & \bar{x}_b &= 0.8, \end{aligned}$$

and the normalized frequency functions

$$\phi(s) = (s + 0.2) \frac{s + 1.5}{s + 0.8}, \quad \Phi(s) = \frac{s + 0.9}{s + 0.4}$$

We found in solving Example 6 that

$$\begin{aligned} K &= v_a/x_a = \bar{v}_b/\bar{x}_b = 5/4 = 1.25, \\ k_a &= v_a/\bar{v}_b = x_a/\bar{x}_b = 5, \\ n_0 &= n_a = 1/\bar{n}_b = 2.5. \end{aligned}$$

First we compute the auxiliary values  $S$  and  $P$ . By Eq. (100)

$$\begin{aligned} S &= \frac{v_a}{k_a} \left[ (n_0 + 1)^2 + k_a (n_0 - 1)^2 \right] \\ &= \frac{5}{5} \left[ 3.5^2 + 5 \times 1.5^2 \right] = 23.5. \end{aligned}$$

By Eq. (104)

$$P = \frac{1 + k_a}{v_a} \quad K = \frac{6}{5} \cdot 1.25 = 1.5.$$

By Eq. (111)

$$\begin{aligned}
 x_{1,4} &= \frac{S}{2n_0} \left[ 1 \pm \sqrt{1 - \frac{4Kn_0^2}{SP}} \right] = \frac{23.5}{5} \left[ 1 \pm \sqrt{1 - \frac{31.25}{35.25}} \right] \\
 &= 4.7 \left[ 1 \pm \sqrt{1 - 0.886525} \right] = 4.7 \left[ 1 \pm \sqrt{0.113475} \right] \\
 &= 4.7 \left[ 1 \pm 0.336861 \right] .
 \end{aligned}$$

Hence,

$$x_1 = 6.283247, \quad x_4 = 3.116753 .$$

We now compute

$$\left| \frac{n_0 + 1}{n_0 - 1} \right| = 3.5/1.5 = 2.333333 < 5 = k_a .$$

The fact that  $k_a = 2.333333$  decides about the ambiguity in choosing  $x_{2,3}$  according to Eqs. (112a, b). Thus

$$\begin{aligned}
 x_{2,3} &= \frac{S}{2K} \left[ 1 \pm \sqrt{1 - \frac{4K}{SP}} \right] = \frac{23.5}{2.25} \left[ 1 \pm \sqrt{1 - \frac{5}{35.25}} \right] \\
 &= 9.4 \left[ 1 \pm \sqrt{1 - 0.141844} \right] = 9.4 \left[ 1 \pm \sqrt{0.858156} \right] \\
 &= 9.4 \left[ 1 \pm 0.926367 \right] .
 \end{aligned}$$

Hence,

$$x_2 = 18.107850, \quad x_3 = 0.692150 .$$

As a check, let us compute the elements of the chain matrix of the lattice. Since it is supposed to be equivalent with the tandems obtained in Example 6, we must obtain the same matrix elements as there. By Eqs. (76, . . . , e) we obtain:

$$\begin{aligned}
 \hat{A}_x &= \phi^2(s) + \frac{12.533348}{19.583329} \Phi^2(s) + \frac{113.776094 + 2.157261}{19.583329} \phi(s) \cdot \Phi(s) \\
 &= \phi^2(s) + 0.64 \Phi^2(s) + 5.92 \phi(s) \cdot \Phi(s) = \hat{A}_{AC} , \quad \text{o. k.}
 \end{aligned}$$

$$\begin{aligned}\hat{B}_x &= 18.8 \left| \phi(s) + 0.64 \frac{9.4}{18.8} \Phi(s) \right| \phi(s) \cdot \Phi(s) \\ &= 18.8 \left| \phi(s) + 0.32 \Phi(s) \right| \phi(s) \cdot \Phi(s) = \hat{B}_{AC}, \text{ o. k.}\end{aligned}$$

$$\begin{aligned}\hat{C}_x &= \frac{1}{19.583329} \left| 9.4 \phi(s) + 18.8 \Phi(s) \right| \\ &= 0.48 \phi(s) + 0.96 \Phi(s) = \hat{C}_{AC}, \text{ o. k.}\end{aligned}$$

$$\begin{aligned}\hat{D}_x &= \phi^2(s) + 0.64 \Phi^2(s) + \frac{4.343949 + 56.437696}{19.583329} \phi(s) \cdot \Phi(s) \\ &= \phi^2(s) + 0.64 \Phi^2(s) + 3.104 \phi(s) \cdot \Phi(s) = \hat{D}_{AC}, \text{ o. k.}\end{aligned}$$

$$\hat{E}_x = 0.64 \Phi^2(s) - \phi^2(s) = \hat{E}_{AC}, \text{ o. k.}$$

Our results have thus been proved to be correct.

In accordance with Figure 9 the lattice has the following branch impedances:

$$X_1 = x_1 \cdot \phi(s) = 6.283247 (s + 0.2) \frac{s + 1.5}{s + 0.8},$$

$$X_4 = x_4 \cdot \phi(s) = 3.116753 (s + 0.2) \frac{s + 1.5}{s + 0.8},$$

$$X_2 = x_2 \cdot \Phi(s) = 18.107350 \frac{s + 0.9}{s + 0.8}$$

$$X_3 = x_3 \cdot \Phi(s) = 0.692150 \frac{s + 0.9}{s + 0.4}.$$

In Example 6 we computed the elements of the circuit shown in Figure 33 that has the impedance function  $\phi(s)$  and the elements of the circuit shown in Figure 34 that has the impedance function  $\Phi(s)$ . Thus, we have only to multiply the inductances and the resistances and to divide the capacitances in those circuits by the values  $x_1, \dots, x_4$  to obtain the elements of the lattice that is presented in Figure 37.

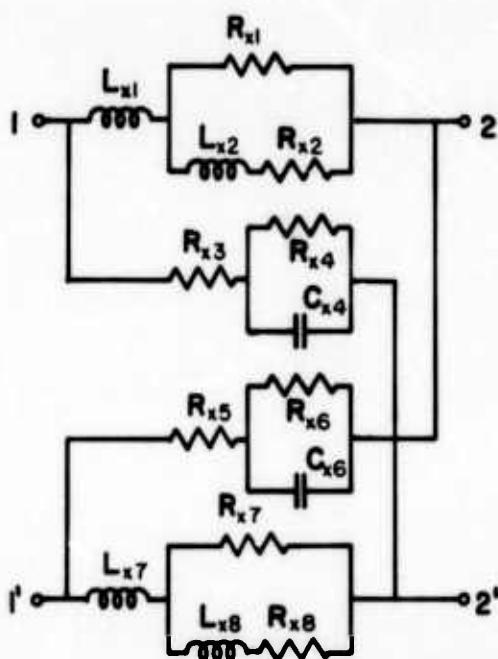


Figure 37. Lattice Two-Port Equivalent to the Tandems in Figures 35 and 36

With	$L_{01} = 1$ ,	$R_{01} = 0.9$ ,
	$L_{02} = 1.928572$ ,	$R_{02} = 0.642858$ ,
		$R_{03} = 1$ ,
	$C_{04} = 2$ ,	$R_{04} = 1.25$ .

from Example 6 we obtain:

$L_{x1} = x_1 L_{01} = 6.283247$	$R_{x1} = x_1 R_{01} = 5.654922$
$L_{x2} = x_1 L_{02} = 12.117694$	$R_{x2} = x_1 R_{02} = 4.039236$
	$R_{x3} = x_2 R_{03} = 18.107850$
$C_{x4} = C_{04}/x_2 = 0.110449$	$R_{x4} = x_2 R_{04} = 22.634813$
	$R_{x5} = x_3 R_{03} = 0.692150$
$C_{x6} = C_{04}/x_3 = 2.889547$	$R_{x6} = x_3 R_{04} = 0.865188$
$L_{x7} = x_4 L_{01} = 3.116753$	$R_{x7} = x_4 R_{01} = 2.805078$
$L_{x8} = x_4 L_{02} = 6.010883$	$R_{x8} = x_4 R_{02} = 2.003630$ .

### 8. EXAMPLE 8 (referring to Section 6)

#### 8.1 Problem

In Example 5 we have computed a tandem that is shown in Figure 31; its circuit elements are presented there. The tandem is matched, but not perfectly matched. What port impedance has to be added in order to obtain a perfectly matched tandem? What is the equivalent and terminated lattice two-port?

#### 8.2 Solution

As it has been mentioned in Example 5, the tandem shown in Figure 7 is a block diagram of the tandem shown in Figure 31. It is a tandem such that its half on the left side is a pcsa T of type (A), since  $n_a = 8 > 1$ , and its half on the right side is of type (C). With  $\bar{n}_b = 2.5$ , we obtain the product

$$n_a \bar{n}_b = 20 > 1.$$

According to Theorem 5 the tandem can only be transformed into a perfectly matched tandem with an equivalent lattice by a forward transposition of port impedance. Hence, it is necessary that we add an in-port impedance  $X_s$ . The port impedance must imply the frequency function

$$\Phi(s) = \frac{1}{s + 0.6}.$$

A block diagram of the tandem with the added in-port impedance is the left side portion of Figure 10 with the upper line of the relation between the ratio coefficients.

The necessary in-port impedance  $X_s = x_s \cdot \Phi(s)$  is obtained by Eq. (132) that gives

$$\begin{aligned} x_s &= \frac{1}{n_a \bar{n}_b - 1} \left[ x_a (n_a - 1)^2 + \bar{x}_b n_a^2 (\bar{n}_b + 1)^2 \right] \\ &= \frac{1}{19} 5 \times 49 + \frac{10}{3} 64 \times 12.25 = \frac{735 + 7840}{19 \times 3} = \frac{8575}{57} = 150.438596. \end{aligned}$$

This in-port impedance will now be totally transposed to the out-port as shown at the right in Figure 10. The primed constants are obtained by the following equations:

By Eq. (121)

$$n'_a = n_a \frac{x_s}{x_s + x_a(n_a - 1)^2} = 8 \frac{150.438596}{150.438596 + 5 \times 49}$$

$$= \frac{1203.508768}{395.438596} = 3.043478 .$$

By Eq. (122)

$$x''_s = \frac{x_s}{n_a n_a} = \frac{150.438596}{24.347824} = 6.178729 .$$

By Eq. (123)

$$\frac{1}{x'_a} = \frac{1}{x_a} + \frac{1}{x''_s} - \frac{1}{x''_s} = 0.2 + 0.006647 - 0.161846 = 0.044801 = 1/22.320930 .$$

By Eq. (124)

$$v'_a = v_a \frac{x'_a}{x_a} = \frac{66.962790}{5} = 13.392558 .$$

By Eq. (125)

$$\bar{n}'_b = \bar{n}_b \frac{x''_s}{x''_s + \bar{x}_b(\bar{n}_b + 1)^2} = 2.5 \frac{6.178729}{6.178729 + 122.5/3}$$

$$= \frac{15.446823}{47.012062} = 0.328571 = 1/3.043482 , \text{ o.k.} = 1/n'_a .$$

By Eq. (126)

$$x''_s = \frac{x''_s}{\bar{n}_b \bar{n}'_b} = \frac{6.178729}{0.821428} = 7.521936 .$$

By Eq. (127)

$$\frac{1}{x'_b} = \frac{1}{x_b} + \frac{1}{x''_s} - \frac{1}{x''_s} = 0.3 + 0.161846 - 0.132944$$

$$= 0.328902 = 1/3.040419 .$$

By Eq. (128)

$$\bar{v}'_b = \bar{v}_b \frac{\bar{x}'_b}{\bar{x}_b} = \frac{6.080838 \times 3}{10} = 1.824251$$

All the constants of the perfectly matched tandem are now known. The tandem is terminated at the out-port by the impedance

$$X'_s = x'_s \Phi(s) = 7.521936 (s + 0.6)$$

We are not interested with its circuitry, since we replace the perfectly matched tandem by its lattice equivalence. This circuit is shown in Figure 38 both as a block diagram and a circuit diagram.

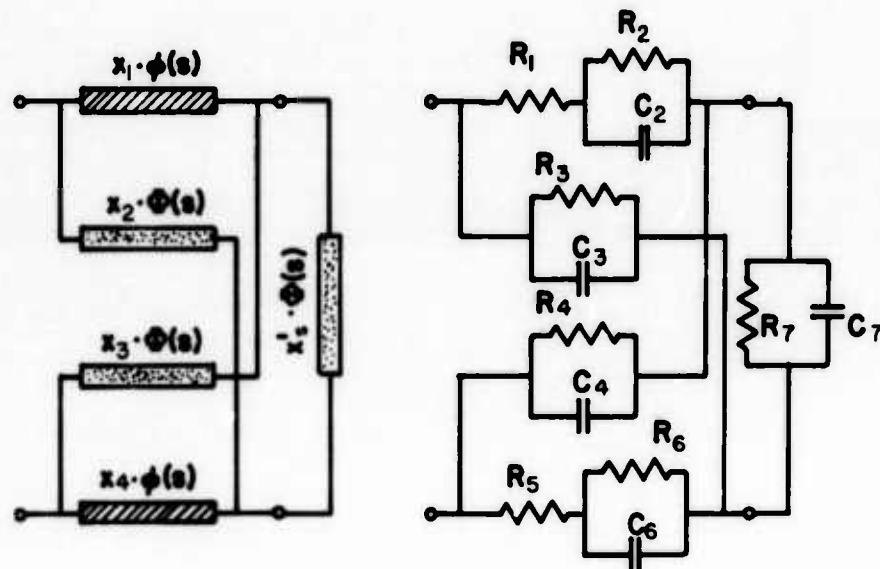


Figure 38. Lattice With Out-Port Termination

The constants of the perfectly matched tandem which we just computed are:

Left Side pcsa T Type (A)	Right Side pcsa T Type (C)
$n'_a = 3.043478$	$\bar{n}'_b = 0.328571$
$v'_a = 13.392558$	$\bar{v}'_b = 1.824251$
$x'_a = 22.320930$	$x'_b = 3.040419$

Therefore, we obtain the series branches

$$u'_a = 27.367397$$

$$w'_a = -8.992152$$

$$u'_b = 2.423647$$

$$w'_b = 7.376319$$

The termination constant is

$$x'_s = 7.521936$$

By Eq. (47)

$$K = \frac{v'_a}{x'_a} = \frac{13.392558}{22.320930} = 0.6$$

by Eq. (48)

$$k_a = \frac{v'_a}{v'_b} = \frac{13.392558}{1.824251} = 7.341400$$

With  $n_0 = n'_a = 3.043478$  we obtain the auxiliary values  $S$  and  $P$  by Eqs. (100) and (104):

$$S = \frac{v'_a}{k_a} \left[ (n_0 + 1)^2 + k_a (n_0 - 1)^2 \right]$$

$$\begin{aligned} S &= 1.824251 \left[ 4.043478^2 + 7.341400 \times 2.043478^2 \right] \\ &= 1.824251 \left[ 16.349714 + 7.341400 \times 4.175802 \right] \\ &= 1.824251 \times 47.005947 = 85.750646 \end{aligned}$$

$$P = \frac{1 + k_a}{v'_a} K = \frac{8.341400}{13.392558} 0.6 = \frac{5.004840}{13.392558} = 0.373703$$

Hence,

$$SP = 32.045274$$

By Eq. (111)

$$\begin{aligned}
 x_{1,4} &= \frac{S}{2n_0} \left[ 1 \pm \sqrt{1 - \frac{4Kn_0^2}{SP}} \right] = \frac{85.750646}{6.086956} \left[ 1 \pm \sqrt{1 - \frac{22.230619}{32.045274}} \right] \\
 &= 14.087607 \left[ 1 \pm \sqrt{1 - 0.693725} \right] = 14.087607 \left[ 1 \pm \sqrt{0.306275} \right] \\
 &= 14.087607 \left[ 1 \pm 0.553421 \right] .
 \end{aligned}$$

Hence, by Eq. (110a)

$$x_1 = 14.087607 \times 1.553421 = 21.883985 ,$$

and by Eq. (110b)

$$x_4 = 14.087607 \times 0.446579 = 6.291229 .$$

We compute

$$\left| \frac{n_0 + 1}{n_0 - 1} \right| = \frac{4.043478}{2.043478} = 1.978724 < k_a = 7.341400 .$$

Therefore,

$$x_{2,3} = \frac{S}{2K} \left[ 1 \pm \sqrt{1 - \frac{4K}{SP}} \right]$$

with index 2 referring to the + sign.

$$\begin{aligned}
 x_{2,3} &= \frac{85.750646}{1.2} \left[ 1 \pm \sqrt{1 - \frac{2.4}{32.045274}} \right] \\
 &= 71.458872 \left[ 1 \pm \sqrt{1 - 0.074894} \right] \\
 &= 71.458872 \left[ 1 \pm \sqrt{0.925106} \right] = 71.458872 \left[ 1 \pm 0.961824 \right] .
 \end{aligned}$$

By Eqs. (112a, b)

$$x_2 = 71.458872 \times 1.961824 = 140.189730$$

$$x_3 = 71.458872 \times 0.038176 = 2.728014 .$$

The impedances  $X_1$  and  $X_4$  imply the normalized frequency function  $\phi(s)$ , the impedances  $X_2$ ,  $X_3$ , and  $X_s$  imply the function  $\Phi(s)$ . We have shown in Example 5 the circuits that have these functions as impedance functions; they are presented in Figures 29 and 30 with the element magnitudes noted there. Hence, we are now able to compute the elements of the lattice circuit shown at the right of Figure 38. They are:

$$\begin{array}{ll}
 R_1 = x_1 R_{01} = 21.883985 & C_2 = C_{02}/x_1 = 0.057119 \\
 R_2 = x_1 R_{02} = 43.767970 & C_3 = C_0/x_2 = 0.007133 \\
 R_3 = x_2 R_0 = 233.649550 & C_4 = C_0/x_3 = 0.366567 \\
 R_4 = x_3 R_0 = 4.546690 & C_6 = C_{02}/x_4 = 0.198689 \\
 R_5 = x_4 R_{01} = 6.291229 & C_7 = C_0/x'_s = 0.132944 \\
 R_6 = x_4 R_{02} = 12.582458 & \\
 R_7 = x'_s R_0 = 12.536560 & 
 \end{array}$$

## 9. EXAMPLE 9 (referring to Section 8)

### 9.1 Problem

Assume a pcsa T of type (A) that is terminated by the impedance  $Z_t(s) = 1$ ; thus the termination is the normalized resistance. Let the pcsa T have the constants

$$n = 2.5, \quad v = 2, \quad x = 3;$$

let

$$\phi(s) = s + 0.8,$$

and

$$\Phi(s) = 1/\phi(s) = \frac{1}{s + 0.8}.$$

A block diagram of the terminated pcsa T is shown in Figure 39.

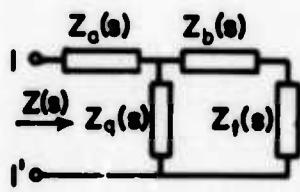


Figure 39. Terminated  
T of Example 9

$$\begin{aligned}
 Z_a(s) &= v(n-1) \phi(s) = 3(s+0.8) , \\
 Z_b(s) &= v \left( \frac{1}{n} - 1 \right) \phi(s) = -1.2(s+0.8) , \\
 Z_q(s) &= v \cdot \phi(s) + x/\phi(s) \\
 &= 2 \frac{s^2 + 1.6s + 2.14}{s+0.8} \\
 Z_t(s) &= 1 .
 \end{aligned}$$

What is the driving-point impedance measured at the in-port terminals? Realize this driving-point impedance by a circuit that is shown by its block diagram in Figure 40.

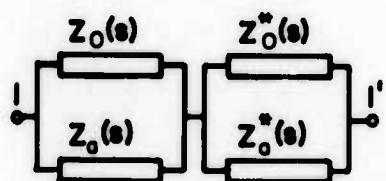


Figure 40. Block Diagram  
of the Realization of  $Z(s)$   
in Example 9

$$\begin{aligned}
 \text{Let } Z_a(s) \cdot Z_a^*(s) &= T^2 , \\
 Z_0(s) \cdot Z_0^*(s) &= T^2 ,
 \end{aligned}$$

with the duality constant  $T$  to be determined.

## 9.2 Solution

The driving-point impedance according to Figure 39 is

$$\begin{aligned}
 Z(s) &= Z_a(s) + \frac{1}{\frac{1}{Z_q(s)} + \frac{1}{Z_b(s) + Z_t(s)}} \\
 Z(s) &= 3s + 2.4 + \frac{1}{\frac{s+0.8}{2s^2 + 3.2s + 4.28} - \frac{1}{1.2s - 0.04}} \\
 &= 6.25 \frac{s^2 + 2.68s + 2.104}{s^2 + 2.85s + 5.39} .
 \end{aligned}$$

Note that  $Z(s)$  is an impedance function in which the degrees of the numerator and of the denominator polynomials are 2 and for which

$$\left( \sqrt{5.39} - \sqrt{2.104} \right)^2 < 2.68 \times 2.85$$

is true. Thus  $Z(s)$  is a positive real function. When we would intend to realize  $Z(s)$  in the classical Brune fashion, it would be necessary to derive at first the minimum resistance function

$$Z(s) = Z_k \frac{s^2 + a_1 s + a_0}{s^2 + b_1 s + b_0}$$

with a constant  $Z_k$  that has impedance character and for which

$$(\sqrt{a_0} - \sqrt{b_0})^2 = a_1 b_1 .$$

Using the general pcsa T instead of the pcsa T with  $\phi_0(s) = 1/\Phi_0(s) = s$ , the impedance function  $Z(s)$  can be realized immediately.

We are now going to realize

$$Z(s) = 6.25 \frac{s^2 + 2.68s + 2.104}{s^2 + 2.85s + 5.39}$$

by the circuit shown in Figure 40 in its block diagram. This circuit has the driving-point impedance given by Eq. (184). In this equation the impedance  $Z_0(s)$  as well as the duality constant  $T$  are unknown. In Section 8 we postulated that the dual impedance function  $Z_0^*(s) = T^2/Z_0(s)$  should have the same zeros as the shunt impedance  $Z_q(s)$  of the pcsa T. Thus with positive and real constants  $T$  and  $K$  and with denominator coefficients  $a_1$  and  $a_0$  all unknown

$$Z_0^*(s) = \frac{T^2}{K} \frac{s^2 + 1.6s + 2.14}{s^2 + a_1 s + a_0} .$$

Hence,

$$Z_0(s) = T^2/Z_0^*(s) = K \frac{s^2 + a_1 s + a_0}{s^2 + 1.6s + 2.14} .$$

We evaluate at  $s = \infty$

$$Z_0(\infty) = K$$

and

$$Z(\infty) = 6.25 .$$

At this frequency  $s = \infty$

$$Z_a(\infty) = \infty, \text{ and } Z_a^*(\infty) = 0 .$$

Therefore, at  $s = \infty$ , according to Figure 40, the parallel circuit at the right in the block diagram is short circuited and the parallel circuit at the left in the diagram degenerates to  $Z_0(\infty)$  since  $Z_a(\infty)$  is open circuit; at the terminals of the complete circuit, therefore, we measure  $Z(\infty) = Z_0(\infty)$ , and therefore

$$K = 6.25 .$$

Hence,

$$Z_0(s) = 6.25 \frac{s^2 + a_1 s + a_0}{s^2 + 1.6s + 2.14} ,$$

in which formula  $a_1$  and  $a_0$  are positive real coefficients, so far still unknown.

Let us now consider Eq. (184). According to this equation

$$Z(s) = \frac{Z_a(s) \cdot Z_0(s) + T^2}{Z_a(s) + Z_0(s)} .$$

By the block diagram shown in Figure 39 we find that not only when  $Z_q(s) = 0$  do we measure  $Z_a(s)$  as the driving-point impedance of the circuit, but also when  $A_b(s) + Z_t(s) = 0$ . Since  $Z_b(s)$  is a negative impedance and  $Z_t(s)$  is a positive one, there is certainly a positive and real  $s = k$  for which

$$Z(k) = Z_a(k) .$$

Evaluating at this frequency Eq. (184) becomes

$$Z_a(k) = \frac{Z_a(k) \cdot Z_0(k) + T^2}{Z_a(k) + Z_0(k)} ,$$

or

$$Z_a^2(k) + Z_a(k) \cdot Z_0(k) = Z_a(k) \cdot Z_0(k) + T^2 .$$

Hence,

$$T^2 = Z_a^2(k) .$$

or, since  $T$  is assumed to be positive and real

$$T = Z_a(k) .$$

The frequency  $k$  is a positive and real solution of the equation

$$Z_b(s) + Z_t(s) = 0$$

or

$$1.2s - 0.04 = 0 .$$

Hence,

$$k = 0.04/1.2 = 1/30 .$$

Therefore,

$$T = Z_a(1/30) = 0.1 + 2.4 = 2.5 , \quad T^2 = 6.25 .$$

The inversion of Eq. (184) is given in Eq. (186) by which we are now able to determine completely

$$\begin{aligned} Z_0(s) &= \frac{T^2 - Z_a(s) \cdot Z(s)}{Z(s) - Z_a(s)} \\ &= \frac{6.25(s^2 + 2.85s + 5.39) - 6.25(3s + 2.4)(s^2 + 2.68s + 2.104)}{6.25(s^2 + 2.68s + 2.104) - (3s + 2.4)(s^2 + 2.85s + 5.39)} \\ &= \frac{s^2 + 2.85s + 5.39 - (3s + 2.4)(s^2 + 2.68s + 2.104)}{s^2 + 2.68s + 2.104 - (0.48s + 0.384)(s^2 + 2.85s + 5.39)} \end{aligned}$$

(equation continued)

$$\begin{aligned}
 &= \frac{3s^3 + 9.44s^2 + 9.894s - 0.3404}{0.48s^3 + 0.752s^2 + 1.0016s - 0.03424} \\
 &= \frac{(s - 1/30)(3s^2 + 9.54s + 10.212)}{(s - 1/30)(0.48s^2 + 0.768s + 1.0272)} .
 \end{aligned}$$

Hence,

$$\begin{aligned}
 Z_0(s) &= 6.25 \frac{s^2 + 3.18s + 3.404}{s^2 + 1.6s + 2.14} , \\
 Z_0^*(s) &= T^2/Z_0(s) = \frac{s^2 + 1.6s + 2.14}{s^2 + 3.18s + 3.404} .
 \end{aligned}$$

As we postulated,  $Z_0^*(s)$  has the same zeros (the same numerator) as  $Z_q(s)$ , but it has a different denominator and another constant of impedance character ahead of the polynomial fraction.

We will now realize  $Z_0^*(s)$ . Since it has the same zeros as  $Z_q(s)$ , we assume it to be equivalent with a parallel circuit of the impedances  $\rho \cdot Z_q(s)$  and the resistance  $Z_1(s) = 1$ . Thus

$$\begin{aligned}
 \frac{1}{Z_0^*(s)} &= 0.5\rho \frac{s + 0.8}{s^2 + 1.6s + 2.14} + 1 \\
 &= \frac{s^2 + (1.6 + 0.5\rho)s + 2.14 + 0.4\rho}{s^2 + 1.6s + 2.14} \\
 &= \frac{s^2 + 3.18s + 3.404}{s^2 + 1.6s + 2.14} .
 \end{aligned}$$

By comparing the coefficients we find that

$$1.6 + 0.5\rho = 3.18 \longrightarrow \rho = 3.16 ,$$

$$2.14 + 0.4\rho = 3.404 \longrightarrow \rho = 3.16 .$$

Hence, the anticipated realization is possible. The realization of

$$Z_q(s) = v \cdot \phi(s) + x/\phi(s) = (2s + 1.6) + \frac{3}{s + 0.8}$$

is shown in Figure 41.

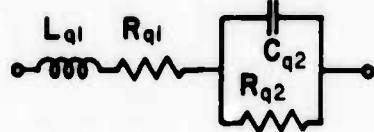
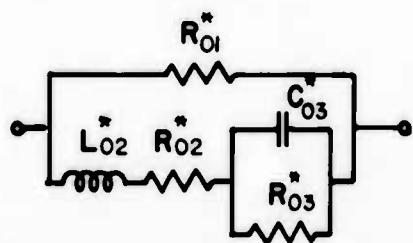


Figure 41. Shunt Realization

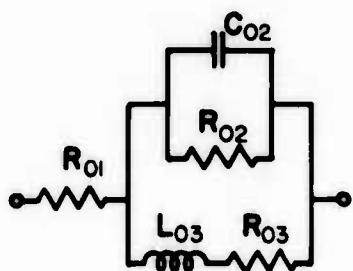
$$\begin{array}{ll} L_{q1} = 2 & R_{q1} = 1.6 \\ C_{q2} = 1/3 & R_{q2} = 3/0.8 = 3.75 \end{array}$$

The realization of  $Z_0^*(s)$  is shown in Figure 42. The circuit implies a resistance  $R_{01}^* = 1$  in parallel with the circuit given by  $Z_q(s)/\rho$ , since  $\rho$  has been a factor by which we multiplied the admittance  $1/Z_0^*(s)$ .

Figure 42. Realization of the Impedance  $Z_0^*(s)$ 

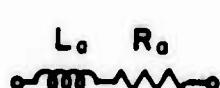
$$\begin{array}{ll} R_{01}^* = 1 & L_{02}^* = L_{q1}/\rho = 0.632911 \\ R_{02}^* = R_{q1}/\rho = 0.506329 & \\ R_{03}^* = R_{q2}/\rho = 1.186709 & C_{03}^* = C_{q2} \cdot \rho = 3.16/3 \end{array}$$

By the well known rules of duality we find the elements of the impedance  $Z_0(s)$  that is dual to  $Z_0^*(s)$ . The circuit of this impedance is shown in Figure 43.

Figure 43. Realization of the Impedance  $Z_0(s)$ 

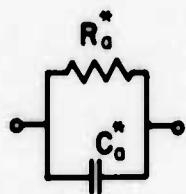
$$\begin{array}{ll} R_{01} = T^2/R_{01}^* = 6.25 & C_{02} = L_{02}^*/T^2 \\ R_{02} = T^2/R_{02}^* = 12.343753 & = 0.101266 \\ R_{03} = T^2/R_{03}^* = 5.266667 & L_{03} = C_{03}^* \cdot T^2 \\ & = 6.583333 \end{array}$$

The circuit realizing the impedance function  $Z_a(s)$  is shown in Figure 44.



$$L_a = 3, \quad R_a = 2.4.$$

Figure 44. Realization of the Impedance  $Z_a(s)$



The circuit realizing the impedance function  $Z_a^*(s) = T^2/Z_a(s)$  is shown in Figure 45.

$$C_a^* = L_a/T^2 = 0.48 ,$$

$$R_a^* = T^2/R_a = 2.604167 .$$

Figure 45. Realization of the Impedance  $Z_a^*(s)$

The complete circuit by which the function  $Z(s)$  is realized is shown in Figure 46.

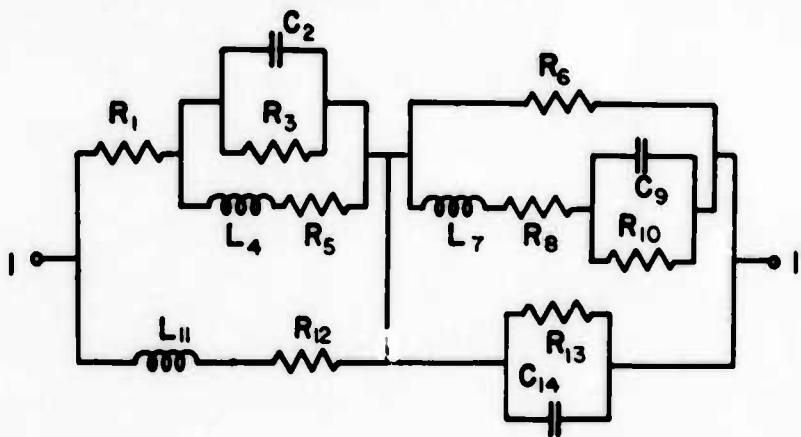


Figure 46. Circuit Realization of  $Z(s)$  in Example 9

$$R_1 = R_{01} = 6.25$$

$$C_2 = C_{02} = 0.101266$$

$$R_3 = R_{02} = 12.343753$$

$$L_4 = L_{03} = 6.583333$$

$$R_5 = R_{03} = 5.266667$$

$$R_6 = R_{01}^* = 1 ,$$

$$L_7 = L_{02}^* = 0.632911$$

$$R_8 = R_{02}^* = 0.506329$$

$$C_9 = C_{03}^* = 3.16/3$$

$$R_{10} = R_{03}^* = 1.186709$$

$$L_{11} = L_a = 3$$

$$R_{12} = R_a = 2.4$$

$$R_{13} = R_a^* = 2.604167$$

$$C_{14} = C_a^* = 0.48 .$$

At this point we would like to perform a parity check on the circuit shown in Figure 46. We compute

$$Z(1) = 6.25 \frac{5.784}{9.24} = 3.912338 .$$

Figure 47(a) is a block diagram of the circuit shown in Figure 46. Into the block we have written the impedance evaluations for  $s = 1$  for each branch element of the circuit in Figure 46. Thus, each numerical value in the blocks is an impedance. First we combined all series and parallel blocks obtaining the diagram in part (b) of Figure 47. We advanced in the same way and obtained the block diagram in part (c) of the figure. Then we obtained the diagram in part (d) and finally the block in part (e) with a result that is a very good approximation of the expected value  $Z(1) = 3.912338$ . Thus we can trust the results obtained in the computation.

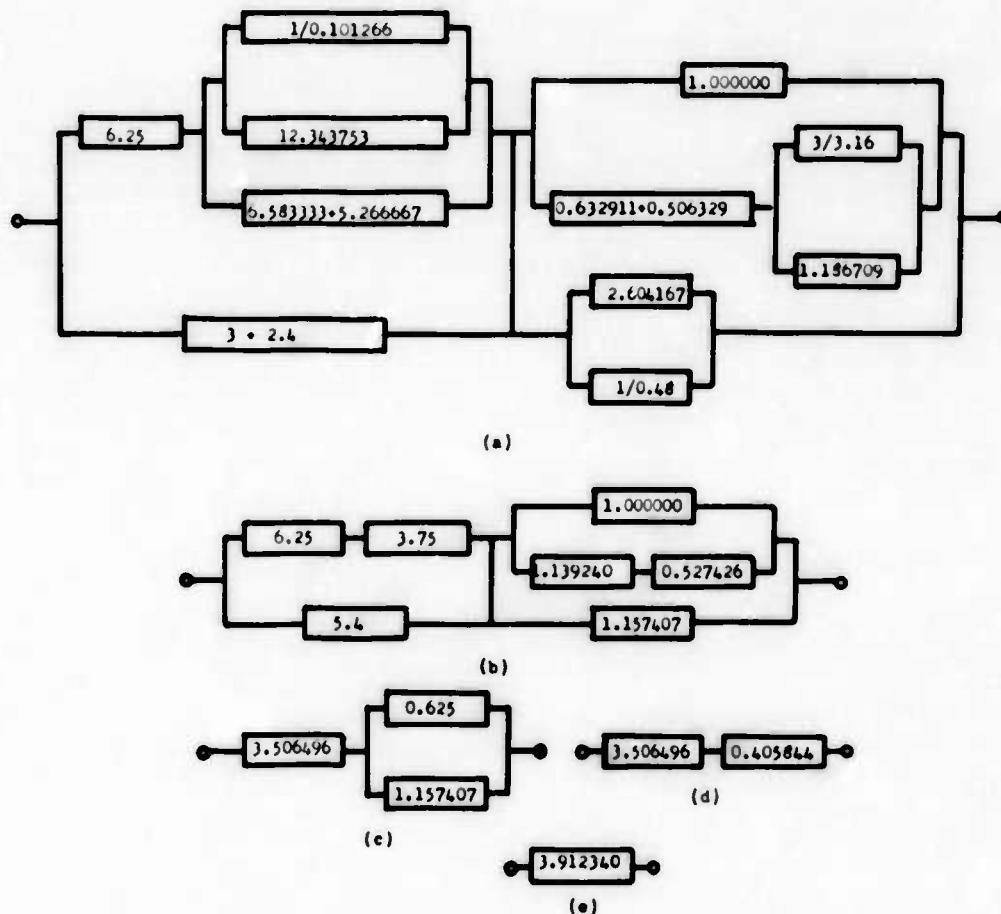


Figure 47. Block Diagrams for Parity Checks

## 10. EXAMPLE 10 (referring to Section 9)

## 10.1 Problem

With the same numerical values of the constants in Example 9 let us determine a perfectly coupled and series-augmented Pi that is terminated by an admittance  $1/Z_t(s) = 1$ . The constants of the pesa Pi are

$$n = 2.5 ,$$

$$v^* = 2$$

$$x^* = 3 ;$$

its normalized frequency functions are

$$\phi(s) = 1/\Phi(s) = s + 0.8 .$$

What is the driving-point admittance of the pesa Pi, what are its branch admittances. Show that the driving-point admittance can be realized by the dual circuit of Figure 46.

## 10.2 Solution

A block diagram of the pesa Pi with its termination is shown in Figure 48. In order to be in strict relation to the duality, we consider the termination as the series combination of the impedance  $Z_t(s) = 1$  and a short circuit connection. The symbol  $\#$  is used in this example to discriminate the values from those in Example 9.

Branch admittances

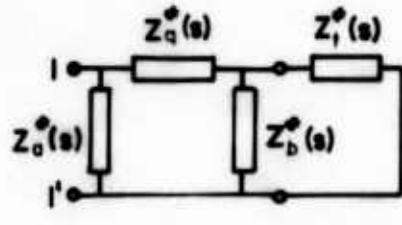


Figure 48. Example of a pesa Pi

$$1/Z_a^\#(s) = v^* (n-1) \phi(s) = 3(s + 0.8) ,$$

$$1/Z_b^\#(s) = v^* \left( \frac{1}{n} - 1 \right) \phi(s) \\ = -1.2(s + 0.8) ,$$

$$1/Z_q^\#(s) = v^* \cdot \phi(s) + x^* / \phi(s) \\ = 2 \frac{s^2 + 1.6s + 2.14}{s + 0.8} ,$$

Termination admittance  $1/Z_t^\#(s) = 1$ .

Note that

in Example 9

$$Z_a(s)$$

$$Z_b(s)$$

$$Z_q(s)$$

$$Z_t(s)$$

in Example 10

$$1/Z_a^{\#}(s)$$

$$1/Z_b^{\#}(s)$$

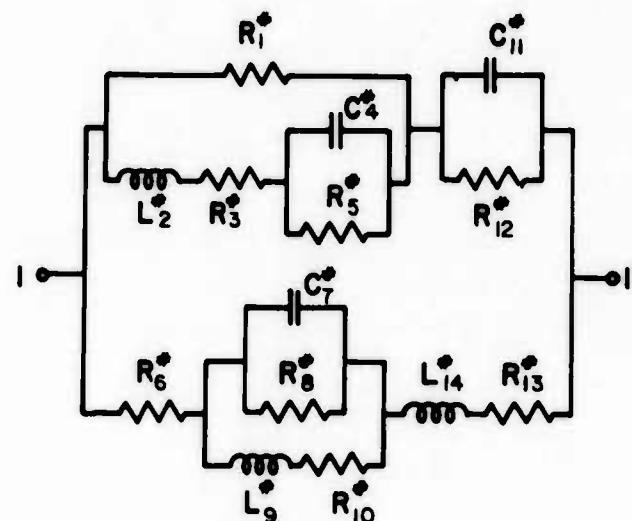
$$1/Z_q^{\#}(s)$$

$$1/Z_t^{\#}(s) + 0$$

The driving-point admittance  $1/Z^{\#}(s)$  according to Figure 48 is

$$\begin{aligned} 1/Z^{\#}(s) &= \frac{1}{Z_a^{\#}(s)} + \frac{1}{Z_q^{\#}(s) + \frac{1}{\frac{1}{Z_b^{\#}(s)} + \frac{1}{Z_t^{\#}(s)}}}, \\ &= 3s + 2.4 + \frac{1}{\frac{1}{2s^2 + 3.2s + 4.28} - \frac{1}{1.2s - 0.04}} \\ &= 6.25 \frac{s^2 + 2.68s + 2.104}{s^2 + 2.85s + 5.39}, \end{aligned}$$

which numerically is the same result as it has been obtained for  $Z(s)$  in Example 9. Therefore  $1/Z^{\#}(s)$  can be realized by the circuit shown in Figure 49 that is the dual of the circuit in Figure 46. A test evaluation for  $s = 1$  that is not performed here shows that the results are correct.



(R, L, C notations without the raised # refer to the circuit in Figure 46 of Example 9)

Figure 49. Circuit Dual to That in Figure 46

$$R_1^{\#} = 1/R_1 = 0.16$$

$$L_2^{\#} = C_2 = 0.101266$$

$$R_3^{\#} = 1/R_3 = 0.081013$$

$$C_4^{\#} = L_4 = 6.583333$$

$$R_5^{\#} = 1/R_5 = 0.189873$$

$$R_6^{\#} = 1/R_6 = 1$$

$$C_7 = L_7 = 0.632911$$

$$R_8^{\#} = 1/R_8 = 1.975000$$

$$L_9^{\#} = C_9 = 3.16/3$$

$$R_{10}^{\#} = 1/R_{10} = 0.842667$$

$$C_{11}^{\#} = L_{11} = 3$$

$$R_{12}^{\#} = 1/R_{12} = 0.416667$$

$$R_{13}^{\#} = 1/R_{13} = 0.384000$$

$$L_{14}^{\#} = C_{14} = 0.48$$

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